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FINANCIAL ECONOMICS

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Contents

List of Figures

List of Tables

Introduction

The sucker has always tried to get something for nothing, and the appeal in all booms is always frankly to the gambling instinct aroused by cupidity and spurred by a pervasive prosperity. People who look for easy money invariably pay for the privilege of proving conclusively that it cannot be found on this sordid earth.
Jesse Livermore

Stylised facts and major puzzles

Stylised Fact

We'll look at more data within each chapter, but there is really one stylised fact that motivates the entire course.

Stylised Fact 1 *Stocks typically earn higher returns to investors than bonds.*

Puzzles

From this stylised fact, we can derive a series of research questions, which we'll label as puzzles, because as a field we still haven't developed convincing and complete answers.

Puzzle 1 *Why do stocks earn higher returns on average than bonds?*

Puzzle 2 *To what extent to market prices reflect all available information?*

Puzzle 3 *Which types of risks offer high returns and which types of risks offer low returns?*

Puzzle 4 *What are the determinants of the choice between equity and debt finance?*

As we will see throughout the course, economists have made some progress on these puzzles, but there is still much to learn. Doing well in this course doesn't necessarily mean knowing the answers to these questions. Instead, doing well means understanding what progress has been made toward answers and understanding what types of methodologies are used by researchers to progress our knowledge and why. At times, this means that we'll spend a lot of effort studying models that do not have a lot of predictive power!

Methodology and learning outcomes

This course requires you to write essays, to prove theorems, to apply mathematical models by hand and with the help of a computer, and to evaluate the efficacy of models both directly and by reviewing the literature. This course is designed to improve your abilities in all of these areas while learning the key introductory concepts in financial economics and asset pricing.

Important takeaways from an education in financial economics

? survey investment managers and financial economics academics to ask, in the wake of the Global Financial Crisis, what traits and skills

Table 1: What firms are seeking and what academics see as the important takeaways

What firms are seeking	Academics' takeaways
Economic reasoning	
Identify the effects of political and environmental developments on an investment portfolio.	Develop a deep understanding of the relationship between risk and return; be aware of the assumptions behind models and the limitations of modelts.
Models and data	
Be able to combine rigorous mathematical analysis with sound economic thinking. Be able to take in large quantities of data, analyse this data and exercise judgment.	Don't be afraid of the math. Be aware that financial theories and models are just heuristics.
Broad knowledge	
Knowledge of history, political economics, philosophy, science and the arts provides tools to critically analyse theories and events.	Read broadly to develop your curiosity; know your history.
Humility	
Be able to admit mistakes and revise opinions and decisions in light of new Information.	Don't be shy to ask questions; be hungry for knowledge; continue to learn

finance students should focus on developing.[1] A summary of their findings is produced in Table 1.

Most of the skills listed here are require application outside the curriculum. What this module is designed to do is give you a set of foundational tools and and insights, focusing on economic reasoning, models and data that will support your learning outside and beyond this module.

[1] Frank J. Fabozzi, Sergio M. Focardi, and Caroline Jonas. *Innvestment Management: A Science to Teach or an Art to Learn?* CFA Institute Research Foundation, 2014. ISBN 978-1-934667-73-6

Computing with Julia

Julia is a new open source programming language designed specifically for scientific computing. The syntax of Julia will be reasonably familiar if you have used other scientific computing languages such as Stata, Matlab, R, Python and Gauss.

We'll use Julia in this course. Julia is reasonably straightforward to use, it is quick, and it is becomming more popular in Financial research as well as in industry. Julia is installed on the student-build computers across the University, and can also be downloaded for free from `https://julialang.org/`, where you will also find many helpful resources to get you started.

Books

Primary text

The core textbook for this course is *The Economics of Financial Markets*, by Roy Bailey[2]. Importantly, Bailey's textbook has a very different style to these course notes, with much greater emphasis on prose explanations of theory and in-depth literature reviews. With my notes alone, you will miss out on a lot of helpful material contained in Bailey's textbook.

[2] Roy Bailey. *The Economics of Financial Markets*. Cambridge University Press, 2005

Other helpful texts

Some of these books cost very little, especially for used copies of early editions. These books will help you attain a greater understanding of the course material, and a higher grade.

- *Investments*, by Bodie, Kane and Marcus.[3]
  Bodie, Kane and Marcus is probably the most popular textbook for advanced undergraduate asset pricing courses. It covers most of the ground that we cover in this course, and much more that we do not cover. Early editions can be purchased very inexpensively from `amazon.co.uk` or `abebooks.co.uk`.

[3] Zvi Bodie, Alex Kane, and Alan J. Marcus. *Investments*. McGraw-Hill, 11 edition, 2017

- *The Econometrics of Financial Markets*, by Campbell, Lo and MacKinlay.[4]
  I have refered to Campbell, Lo and MacKinlay a lot when developing this course. The authors present clear expositions of econometric theory as applied to financial markets as well as thoughtful reviews of the empirical literature. If you are planning to write your dissertation on a finance topic, or to continue to study finance at MSc or PhD level, you should get this book.

[4] John Campbell, Andrew Lo, and A. Craig MacKinlay. *The Econometrics of Financial Markets*. Princeton University Press, 1 edition, 1997

- *Financial Economics*, by Fabozzi, Neave and Zhou.[5]
  This is another useful book that covers much of the same ground as this course, with a nice style (in my opinion).

[5] Frank J. Fabozzi, Edwin H. Neave, and Guofu Zhou. *Financial Economics*. Wiley, 1 edition, 2012

Popular non-fiction

The following books also discuss some of the central themes of this course:

- *Against the Gods: the remarkable story of risk*, by Peter L. Bernstein.[6]
- *A Random Walk Down Wall Street*, by Burton G. Malkiel.[7]

[6] Peter L. Bernstein. *Against the Gods: The Remarkable Story of Risk*. John Wiley & Sons, 1998. ISBN 978-0471295631

[7] Burton G. Malkiel. *A Random Walk Down Wall Street*. W. W. Norton & Company, 1973. ISBN 0-393-06245-7

Roadmap

The course is divided into two distinct parts; the structure mostly formed by considerations relating to the timing of assessment. The first part of the course, weeks 1 to 4 prior to the test, is one topic per lecture. The second part of the course, weeks 6 to 10 following the test, form a more continuous narrative, developing concepts of market efficiency and asset pricing models.

Part 1

Chapter 1 Overview of the course.
Some math revision.
Compound interest and discounting.

Chapter 2 Markets and instruments.
Debt or equity financing.
Private or public financing.

Chapter 3 Fixed income.
Duration and convexity.

Chapter 4 Arbitrage.
Forex and interest rate forwards.

Chapter 5 Asset market efficiency.
The joint hypothesis problem.
Testing for efficiency and the role of asset pricing models.
Event studies.
Computer lab 1 is based on this Chapter.

Chapter 6 The fundamental valuation relationship.
Why do some assets offer higher returns than others?
Stocks vs. insurance.

Chapter 7 Mean variance utility.
What do we lose and gain when we take simple approximations?
Optimal portfolios.
Application: the welfare cost of business cycles.

Chapter 8 Efficient portfolios.
The gains from and limits to diversification.
Mutual fund separation theorems.

Chapter 9 The CAPM.
Can we explain the high returns of stocks?
Application: Network regulation.
Computer lab 2 is based on this Chapter.

Throughout the term, seminars will be used to cover more technical material, and terminal classes will be used to apply the material to real world data. The assessment pattern includes problem sets (15%), coding exercises (15%) and a final exam (70%). The coding exercises are based on lecture and seminar material, and contain questions that are similar in style to the exam. The coding exercises assess the terminal class work, build computational skills and add context to the lecture and seminar material.

1
Principles of Valuation

What could be more interesting than interest rates?
It's right in the name!
Matt Levine, *Money Stuff*, 20 March 2017.

Introduction and overview

This lecture starts by presenting an introduction to the main drivers of the demand for financial assets: investors' beliefs about payoffs, investors' preferences and opportunities for arbitrage. Following this introduction, the lecture provides an overview of some useful statistical results, before an overview of discounting and arbitrage.

Figure 1.1 shows the relative performance of a USD$1 investment in US shares, corporate bonds and money market instruments since 1979. Over this period, shares have outperformed corporate bonds and money market intruments, albeit with significantly more risk. At the start of this period, high inflation rates reduced the real net returns to safe assets and corporate bonds below zero. The return to safe money market instruments has also fallen negative in recent years while the Federal Funds policy interest rate has been set at the zero lower bound. The real returns to shares remained largely positive throughout the period of high inflation at the start of the sample, and despite crashes in 1987, 2000-02 and 2008-09 have delivered high returns over long holding periods.

Figure 1.2 shows the log relative performance of investments in shares and bonds versus safe assets for the past 30 years over a range of holding periods. Over short holding periods, the returns to shares have historically been much risker than for safer corporate bonds or money market instruments (Fed Funds). Over longer holding periods, shares have historically offered predictably higher returns than safer assets.

The main question of this course is to ask where this historical outperformance of shares relative to safer bond and money market investments comes from. Does this historical outperformance reflect past underestimates of returns to shares? If so, we wouldn't expect these patterns to continue indefinitely. Alternatively, does this outperformance form compensation for shareholders for risk? If this latter explanation is true, we should expect shares to continue to outperform bonds over long horizons in future; we should be able to link historical and expected relative performance of shares and bonds to measures of risk and of risk tolerance.

The determination of asset prices

As in any market, asset prices are determined by demand and supply. This course focuses largely on the demand side; what determines the price agents will be willing to pay for a given asset? To a large extent, the demand for financial assets is dependent on arbitrage, beliefs and preferences.

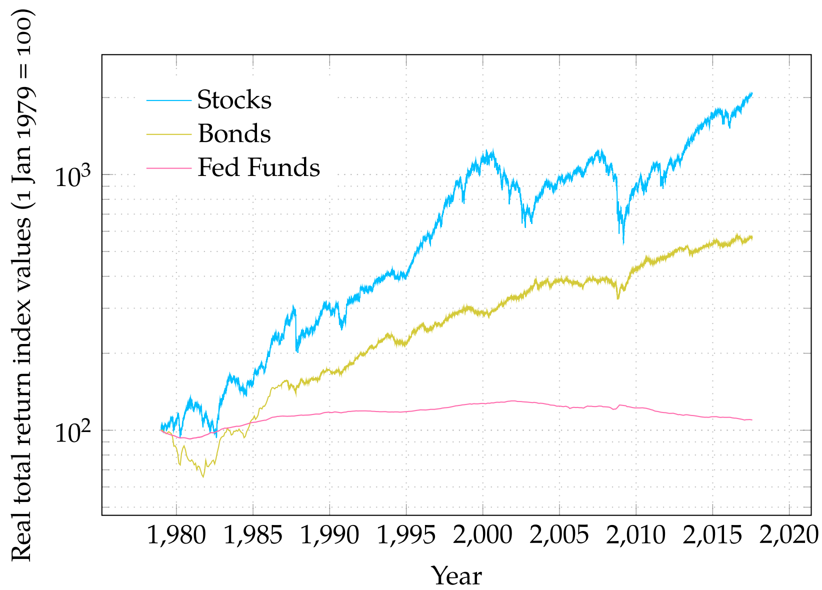


Figure 1.1: The long run performance of shares, corporate bonds and safe assets.

Source: St Louis Federal Reserve, FRED database. (Shares: Russell 3000 Total Return Index, RU3000TR; Bonds: BofA Merill Lynch US Corporate Master Index, BAMLCC0A0CMTRIV; Fed Funds: Effective Federal Funds Rate, DFF; Price Deflator: Consumer Price Index for All Urban Consumers: All Items, CPIAUCSL).

Arbitrage

We can attempt to determine fundamental values of financial assets by appeal to the beliefs and preferences of investors. Where an asset has similar traded alternatives, we can also appeal to arbitrage arguments.

1. The Law of One Price.

The *Law of One Price* tells us that two assets (or portfolios) with the same payoffs should trade at the same price. In finance, it can often be difficult to construct an *identical* portfolio of traded assets that replicates the payoffs of the individual asset we are trying to value. But, we can generally get close to a replicating portfolio, and use the price of the replicating porfolio as a starting point for valuation.

The Law of One Price is maintained by arbitrage, which are trading strategies that require zero initial outlay, and are risk free. Limits to arbitrage would exist when asset markets suffer from transactions costs, and barriers to entry, and also when arbitrageurs cannot finance their trades at the risk free interest rate.

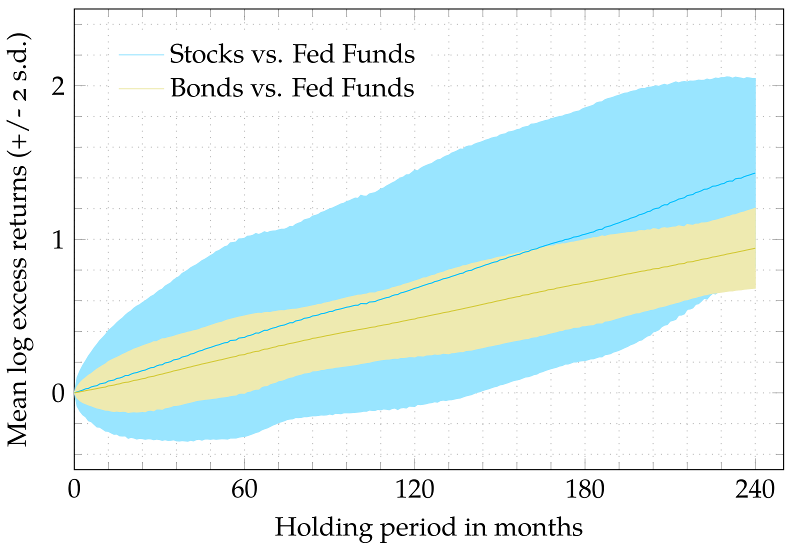


Figure 1.2: Excess returns to risky vs. safe assets

Source: St Louis Federal Reserve, FRED database (Period 1979 - 2017; Shares: Russell 3000 Total Return Index, RU3000TR; Bonds: BofA Merill Lynch US Corporate Master Index, BAMLCC0A0CMTRIV; Fed Funds: Effective Federal Funds Rate, DFF; Price Deflator: Consumer Price Index for All Urban Consumers: All Items, CPIAUCSL).

Beliefs

Financial assets provide uncertain payoffs in future. Prices today reflect *beliefs* held by individuals, which may or may not be consistent with rational expectations. When determining their demand for an asset, investors must form

2. beliefs over the expected payoffs of the asset and
3. beliefs over the risk profile associated with those payoffs.

We would expect assets with higher expected payoffs to have higher prices all else equal. Would we expect assets with higher expected variance to have lower prices all else equal?

Preferences

The second fundamental input into asset prices is the preferences of investors. In particular, we are concerned with

4. investors' time preference and
5. investors' tolerance for risk.

All else equal, we should expect impatient investors to discount assets with longer maturities. Would we expect risk tolerant investors to have a greater demand for risky assets than risk averse investors?

Discounting

Discounting allows us to compare the value of payments occurring at different times. Typically, we would expect assets with returns further in the future to trade at a lower price than assets with payoffs nearer to the present, all else equal. When discounting, we are trying to determine the present value at today's date of a payment occuring in the future.

There are two key approaches two discounting, which form the basis of the field of asset pricing. The first approach is arbitrage: assets with similar payoffs should have similar prices. The second approach combines preferences and beliefs: what is the expected consumption value of the asset, and how does this compare to the opportunity cost of the asset?

In the end, we want to use arbitrage arguments, beliefs and preferences to derive *discount factors* or *discount rates* that can be used to price assets.

Arbitrage

We start with arbitrage, thinking about discounting by appealing to the law of one price: the present value of an asset is determined by the cost of a portfolio that can be purchased today and that replicates the payoffs of the asset. A *replicating portfolio*, if you will.

We start with an example.

Example 1.1 *Suppose there is a bank account that offers an interest rate of 25% for a 5 year deposit, risk free. 100 dollars deposit today returns 125 dollars in five years. What is the value of an asset that pays 100 dollars with certainty in five years?*

Solution 1.1 *Well, if we deposit 80 pounds in the account today, the account will return us 100* $(= 80 \times (1 + 25\%))$ *in five years' time. So, the* present value *of an asset returning 100 dollars in five years is 80 dollars.*

Typically, arbitrage based arguments have high predictive power. Large divergences in prices of identical assets across markets do not typically persist long, much to the annoyance of well-meaning policymakers.[1] Unfortunately arbitrage arguments cannot always be applied. Sometimes it is not possible to construct a perfect replicating portfolio.

[1] See for example this story about Venezuelan official exchange rate arbitrage: `http://www.reuters.com/article/us-venezuela-flights/get-a-boat-venezuela-flights-booked-full-for-months-idUSBRE98N0TW20130924`

Basis risk

Basis risk is the difference between the payoffs of an asset and the payoffs of the replicating portfolio constructed to match that asset. Figure 1.3 provides an example of the type of thing that can go wrong.

Figure 1.3 plots the recent price history of short term light sweet crude oil futures for delivery at Cushing, Oklahoma (West Texas Intermediate) and at London (Brent). The grading standards in terms of purity, sulphur content for the two contracts is very similar, so any differences in price when they exist are *normally* just the reflection of shipping costs, plus some timing issues (the prices are end-of-day, but the futures are traded in different time zones for example).

In 2011 the price of oil in Cushing Oklahoma started to fall below the price in London, quite dramatically. This is largely due to increased shale oil production in the US, which had odd laws preventing the export of oil at the time. When the US was an oil importer, the price at Cushing reflected the world price. As US oil imports fell, the Cushing price fell below the world price.

The West Texas Intermediate and London Brent futures markets provide opportunities for oil producers and consumers to hedge the cost of their consumption, and to construct replicating portfolios of their economic exposure to fluctuations in oil prices. Any oil consumer outside of the US who had hedged their oil cost with West Texas Intermediate futures would have found in 2011 that the value of their hedge had fallen below the cost of their underlying exposure.

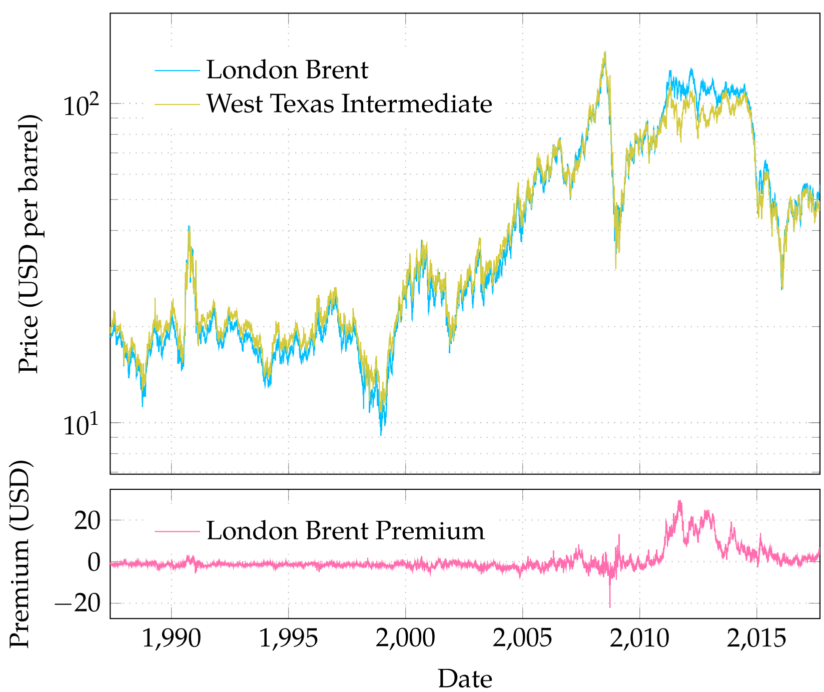


Figure 1.3: Light sweet crude in Cushing and in London.

Source: St Louis Federal Reserve, FRED database and author's calculations. (Brent: London Brent Crude Oil, DCOILBRENTEU; WTI: West Texas Intermediate Crude Oil, DCOILWTICO).

Counterparty risk

Counterparty risk is credit risk inherent in the replicating portfolio. Earlier we considered an example of a bank account offering 25% interest over a five year term. If the bank offering this account defaults, then the replicating portfolio will fail to replicate the 100 dollar payoff of the asset.

Liquidity risk

Some arbitrage strategies require the trader to borrow in order to fund their purchases of assets. This exposes the trader to liquidity risk: if their funding is withdrawn while their trade is still active, they may need to unwind their trade at cost.

Compounding

Compounding is the process of solving for the future value of an asset given the present value and an interest rate (or a process of interest rates over time). We can apply the reverse set of operations to discount the value of future payments and determine the present value of riskless assets.

In this Chapter, we'll quickly look at compounding, before returning to discounting in Chapter 3. We start with an example.

Example 1.2 *Consider two bonds B_1 and B_2, where both bonds pay coupon payments of 6% per annum, but B_1 makes payments annually, while B_2 makes payments semi-annually (that is, bond B_2 makes payments of 3% every six months). Which bond would you prefer to hold?*

Solution 1.2 *Typically, you would prefer to hold bond B_2. Bond B_2 makes more frequent payments, and you can reinvest the proceeds to obtain a greater annual return than bond B_1.*

In sum, the frequency of payments matters, not just the interest rate. In the above example, the two bonds B_1 and B_2 share the *simple rate* of interest of 6%. But, the two bonds have different *effective annual* interest rates, as B_2 returns funds at higher frequency than B_1.

Example 1.3 *Consider an account yielding 6% p.a., compounding semi-annually. This means that the bond pays 3%, each six months. The initial deposit in the account is denoted A_0. What is the amount in the account after one year, $A(1)$?*

Solution 1.3 *After six months:*

$$A(0.5) = A_0\,(1 + 3\%)$$

After one year:

$$\begin{aligned} A(1) &= A(0.5)\,(1+3\%) \\ &= A_0\,(1+3\%)^2 \\ A(1) &= A_0(1.0609) \end{aligned}$$

The effective annual interest rate is 6.09%.

Property 1.1 *The general formula for the effective annual interest rate R of a periodic compounding asset with simple interest rate r, payment frequency of n payments per year is*

$$R = \left(1 + \frac{r}{n}\right)^n$$

Property 1.2 *After t periods, the value of an account with initial balance A_0 with simple rate r, periodicity n is*

$$A(t) = A_0 \left(1 + \frac{r}{n}\right)^{tn}$$

Continuous compounding

When analysing bonds and loans, it makes sense to refer to simple and effective interest rates consistent with the periodic compounding written in the contract. But with other assets, including stocks, returns are not as periodic, but are rather a continuous process. This continuous *force of interest* can be considered by taking the limit of periodic compounding as the compounding period becomes infinitessimally small. This is referred to as *continuous compounding*.

What happens as $n \to \infty$?

Proposition 1.1 *Under continuous compounding ($n \to \infty$)*

$$A(t) = A_0 e^{rt}.$$

Proof. To prove this result, we need to show that

$$\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{tn} = e^{rt}.$$

There are a couple of ways to show this, which are beyond the scope of this course but are useful exercises for students with stronger calculus backgrounds.

■

Arbitrage examples

Example 1.4 *Consider the following foreign exchange rates taken from Yahoo! Finance on 12 August 2016.*

$$\text{GBP/JPY} = 132.26$$

Read as ``GBP 1 is worth JPY 132.26''. This notation is standard but is also an endless source of frustration. Note that to convert GBP 100 *into* JPY, *you would calculate the following:* $100\ \text{GBP} \times \text{GBP}/\text{JPY} = 13226\ \text{JPY}$. *Physicists (rightfully) despair.*

$$\text{GBP}/\text{USD} = 1.2966$$

$$\text{USD}/\text{JPY} = 102.03$$

Is there an arbitrage opportunity available at these prices?

Solution 1.4 *Starting with* USD 1, *purchase* $1 \times \text{USD}/\text{JPY} = \text{JPY}\ 102.03$. *Sell this for* $\text{JPY}\ 102.03/[\text{GBP}/\text{JPY}] = 102.03/132.26 = \text{GBP}\ 0.7714$. *This can be sold for* $\text{GBP}\ 0.7714 \times \text{GBP}/\text{USD} = 0.7714 \times 1.2966 = \text{USD}\ 1.0002$. *This strategy appears to provide a small riskless profit, although this could be a rounding error or the result of relying on mid-prices, which don't necessarily reflect the bid/offer prices available to individual investors.*

Example 1.5 *Consider the following interest rates taken from Bloomberg.com on 12 August 2016. Municipal bonds are issued by US local and state governments. They are generally considered to have very low probability of default, but not as low as US Government bonds issued by the US Federal Government.*

$$\textit{USD Govt Bond 12M Yield} = 0.53\%$$

$$\textit{USD Municipal Bonds 12M Yield} = 0.45\%$$

Why do US municipal bonds offer lower yields than US Government bonds?

Solution 1.5 *The limit to arbitrage in this example is the tax treatment of these two instruments. In the US, retail investors' income from Municipal Bonds is tax-exempt. This lifts the retail demand for Municipal Bonds above that of Federal Government Bonds for retail customers. Note that institutional and foreign investors do not enjoy this differential tax treatment.*

Beliefs and expectations

We model beliefs about payoffs and risk using the *expectation operator*. Behind the scenes, we'll be considering expectations formed with respect to a given information set, that is, $\mathbb{E}(\cdot|\Omega_s)$, where Ω_s is the information set, or *beliefs*, of some investor s. To keep the notation simple, we'll typically just refer $\mathbb{E}(\cdot)$ whenever this is unlikely to cause confusion. The rest of this section is a review of mathematical results that we will rely on in future chapters.

Let Y be a discrete random variable.

State	1	2	3	...	n
Realisation	y_1	y_2	y_3	...	y_n
Probability	π_1	π_2	π_3	...	π_n

The expectation of Y is

$$\mathbb{E}(Y) = \sum_{i=1}^{n} \pi_i y_i$$

Notation: Sometimes we'll use the notation $\mu_Y = \mathbb{E}(Y)$.

Example 1.6 *What is the expected payoff of the following random variable, Y?*

State	1	2	3	4
Realisation	-10	0	6	14
Probability	0.4	0.1	0.3	0.2

Solution 1.6 *To solve for the expectation, we sum over the products of realisations and probabilities associated with each state:*

$$\begin{aligned}
\mathbb{E}(Y) &= \sum_{i=1}^{n} \pi_i y_i \\
&= \pi_1 y_1 + \pi_2 y_2 + \pi_3 y_3 + \pi_4 y_4 \\
&= 0.4 \times (-10) + 0.1 \times 0 + 0.3 \times 6 + 0.2 \times 14 \\
&= -4 + 0 + 1.8 + 2.8 \\
&= 0.6
\end{aligned}$$

The expectation of a function of a random variable

Property 1.3 *The expectation operator is a* linear operator*,*

$$\mathbb{E}(a + bZ) = a + b \cdot \mathbb{E}(Z)$$

Example 1.7 *Rate of return:*

$$r_Y = \frac{Y - P_Y}{P_Y} = \frac{Y}{P_Y} - 1$$

What is the expectation of r_Y?

Solution 1.7

$$\mathbb{E}(r_Y) = \mathbb{E}\left(\frac{Y - P_Y}{P_Y}\right)$$

$$\mathbb{E}(r_Y) = \frac{\mathbb{E}(Y)}{P_Y} - 1$$

Example 1.8 *Asset* $P_Y = 80, y_1 = 120, y_2 = 60, \pi_1 = \pi_2 = 0.5$. *What is the expected return* $\mathbb{E}(r_Y)$*?*

Solution 1.8

$$\begin{aligned}
\mathbb{E}(r_Y) &= \frac{\mathbb{E}(Y)}{P_Y} - 1 \\
&= \frac{0.5 \times 120 + 0.5 \times 60}{80} - 1 \\
&= \frac{90}{80} - 1 \\
&= 12.5\%
\end{aligned}$$

The variance of a random variable

Alongside measures of expected payoffs, we also need to account for investors expectations of risk. The variance operator is a useful starting point.

Definition 1.1 *The variance of random variable* Y *is*

$$\begin{aligned}
var(Y) &= \mathbb{E}[(Y - \mathbb{E}[Y])^2] \\
&= \sum_{i=1}^{n} \pi_i (y_i - \mu_Y)^2
\end{aligned}$$

Notation: Sometimes we'll use the notation $\sigma_Y^2 = \text{var}(Y)$.

Example 1.9 *Consider the variable* Y *from Example 1.6, what is the variance of* Y*?*

Solution 1.9

$$\begin{aligned}
var(Y) &= \mathbb{E}[(Y - \mathbb{E}[Y])^2] \\
&= \sum_{i=1}^{n} \pi_i (y_i - \mu_Y)^2 \\
&= 0.4 \times (-10 - 0.6)^2 + 0.1 \times (0 - 0.6)^2 + 0.3 \times (6 - 0.6)^2 + 0.2 \times (14 - 0.6)^2 \\
&= 0.4 \times 112.36 + 0.1 \times 0.36 + 0.3 \times 29.16 + 0.2 \times 179.56 \\
&= 44.944 + 0.036 + 8.748 + 35.912 \\
&= 89.64
\end{aligned}$$

Property 1.4 *The variance of random variable* Y *can be expressed as follows:*

$$var(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

Proof. Start from the definition of variance,

$$\begin{aligned}
\text{var}(Y) &= \mathbb{E}[(Y - \mathbb{E}[Y])^2] \\
&= \mathbb{E}[Y^2 - 2Y\mathbb{E}(Y) + \mathbb{E}(Y)^2] \\
&= \mathbb{E}[Y^2] - 2\mathbb{E}(Y)\mathbb{E}(Y) + \mathbb{E}(Y)^2 \\
&= \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2
\end{aligned}$$

∎

Example 1.10 *Again, consider the variable* Y *from Example 1.6, calculate the variance of* Y *using Property 1.4.*

Solution 1.10

$$\begin{aligned}
var(Y) &= \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 \\
&= \left(\sum_{i=1}^{n} \pi_i y_i^2\right) - \mu_Y^2 \\
&= (0.4 \times (-10)^2 + 0.1 \times 0^2 + 0.3 \times 6^2 + 0.2 \times 14^2) - 0.6^2 \\
&= 0.4 \times 100 + 0.1 \times 0 + 0.3 \times 36 + 0.2 \times 196 - 0.36 \\
&= 40 + 0 + 10.8 + 39.2 - 0.36 \\
&= 89.64
\end{aligned}$$

It is up to you whether you choose to use Property 1.4 when calculating variances in this course, many students find it easier to do so. *Hint: The variance of a random variable must always be non-negative. If you do get a negative value, it is possible that you've got the order wrong, calculating* $(\mathbb{E}[Y])^2 - \mathbb{E}[Y^2]$ *instead of* $\mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$.

Variance of a function

Property 1.5 *Variance of a function: The general rule for the variance of linear functions of a single random variable is as follows,*

$$var(a + bZ) = b^2 \; var(Y)$$

Proof. Start from the definition of variance given in Definition 1.1.

$$\begin{aligned}
\text{var}(a + bY) &= \mathbb{E}[((a + bY) - \mathbb{E}[a + bY])^2] \\
&= \mathbb{E}[(a + bY - a - b \cdot \mathbb{E}[Y])^2] \\
&= \mathbb{E}[(bY - b \cdot \mathbb{E}[Y])^2] \\
&= b^2 \cdot \mathbb{E}[(Y - \mathbb{E}[Y])^2] \\
&= b^2 \; \text{var(Y)}.
\end{aligned}$$

■

Example 1.11 *The variance of returns. The rate of return for asset Y is given by*

$$r_Y = \frac{Y}{P_Y} - 1.$$

What is the variance of r_Y?

Solution 1.11 *Using Property 1.5, in this case with* $b = \frac{1}{P_Y}$. *Therefore we have*

$$var(r_Y) = \frac{var(Y)}{P_Y^2}.$$

Two random variables

Throughout this course we will often consider cases with two or many random variables. These could be different assets in our portfolio, or alternatively a combination of state-contingent payoffs and marginal utilities. Let $\pi_{ij} = P(X = x_i, Y = y_j)$.

		Value of Y					
		y_1	y_2	y_3	...	y_n	$P(X)$
Value of X	x_1	π_{11}	π_{12}	π_{13}	$\cdots$	π_{1n}	$P(x_1)$
	x_2	π_{21}	π_{22}	π_{23}	$\cdots$	π_{2n}	$P(x_2)$
	x_3	π_{31}	π_{32}	π_{33}	$\cdots$	π_{3n}	$P(x_3)$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
	x_m	π_{m1}	π_{m2}	π_{m3}	$\cdots$	π_{mn}	$P(x_m)$
	$P(Y)$	$P(y_1)$	$P(y_2)$	$P(y_3)$	$\cdots$	$P(y_n)$	

The final row displays the marginal distribution of Y, the final column displays the marginal distribution of X.

Definition 1.2 *The marginal probability of $X = x_i$ is the unconditional probability of the event $X = x_i$ occurring, $P(X = x_i)$.*

Definition 1.3 *The joint probability of the doublet $(X = x_i, Y = y_j)$ is the unconditional probability of both events $X = x_i$ and $Y = y_j$ occurring, $P(X = x_i, Y = y_j)$.*

Definition 1.4 *The conditional probability of the ordered pair $(X = x_i, Y = y_j)$ is the probability of event $X = x_i$ occurring given the known occurrence of $Y = y_i$, $P(X = x_i | Y = y_j)$.*

Example 1.12 *Let there be two discrete random variables, X and Y, described as follows*

		Value of Y		
		-4	0	8
Value of X	3	0.0	0.1	0.4
	2	0.2	0.3	0.0

Find, solve for the following:

a. The marginal probability, $P(X = 3)$

b. The joint probability of the doublet, $P(X = 3, Y = 8)$.

c. The conditional probability, $P(X = 3|Y = 8)$.

Solution 1.12 *a. The marginal probability, $P(X = 3) = 0.5$*

b. The joint probability of the doublet, $P(X = 3, Y = 8) = 0.4$.

c. The conditional probability, $P(X = 3|Y = 8) = 1$.

We can sum across rows and columns to find the marginal distributions of X and Y.

		Value of Y			
		-4	0	8	$P(X)$
Value of X	3	0.0	0.1	0.4	0.5
	2	0.2	0.3	0.0	0.5
	$P(Y)$	0.2	0.4	0.4	

Covariance and correlation

Definition 1.5 *The covariance of X and Y,*

$$cov(X, Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))]$$

Notation: $\sigma_{XY} := cov(X, Y)$.

Property 1.6

$$cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X) \cdot \mathbb{E}(Y)$$

Proof. Start from the definition of covariance:

$$\begin{aligned}
\text{cov}(X, Y) &= \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))] \\
&= \mathbb{E}[XY - X\mathbb{E}(Y) - \mathbb{E}(X)Y + \mathbb{E}(X)\mathbb{E}(Y)] \\
&= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) - \mathbb{E}(X)\mathbb{E}(Y) + \mathbb{E}(X)\mathbb{E}(Y) \\
&= \mathbb{E}(XY) - \mathbb{E}(X) \cdot \mathbb{E}(Y)
\end{aligned}$$

■ Property 1.6 will be very useful in a range of situations. Primarily, terms of the form $\mathbb{E}(XY)$ often emerge in financial economics, and it is useful to be able to decompose these terms into $\text{cov}(X, Y) + \mathbb{E}(X) \cdot \mathbb{E}(Y)$.

Example 1.13 *Back to our example:*

		Value of Y			
		-4	0	8	$P(X)$
Value of X	3	0.0	0.1	0.4	0.5
	2	0.2	0.3	0.0	0.5
	$P(Y)$	0.2	0.4	0.4	

What is the covariance of X and Y?

Solution 1.13

$$\mathbb{E}(X) = 0.5 \times 3 + 0.5 \times 2 = 2.5$$

$$\mathbb{E}(Y) = 0.2 \times (-4) + 0.4 \times 0 + 0.4 \times 8 = 2.4$$

$$\begin{aligned}\mathbb{E}(XY) &= 0.0 \times 3 \times (-4) + 0.1 \times 3 \times 0 + 0.4 \times 3 \times 8 \\ &\quad + 0.2 \times 2 \times (-4) + 0.3 \times 2 \times 0 + 0.0 \times 2 \times 8 \\ &= 8\end{aligned}$$

$$\begin{aligned}cov(X, Y) &= \mathbb{E}(XY) - \mathbb{E}(X) \cdot \mathbb{E}(Y) \\ &= 8 - 2.5 \times 2.4 \\ &= 2\end{aligned}$$

Property 1.7 *The covariance of a random variable Y with itself is equal to the variance of Y,*

$$cov(Y, Y) = var(Y)$$

Property 1.8 *The covariance operator is* commutative*,*

$$cov(X, Y) = cov(Y, X)$$

Definition 1.6 *Correlation,*

$$corr(X, Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Sometimes, we'll use the notation $\rho_{XY} := corr(X, Y)$.
Note that correlation is a normalised measure, $\rho_{XY} \in [-1, 1], \quad \forall\ X, Y$.

Functions of two random variables

Suppose X and Y are payoffs from two investments. W represents the payoff from a portfolio which holds both investments X and Y. The parameters a and b are the amounts invested in investments X and Y respectively.

$$W = aX + bY$$

Property 1.9 *The expectation of a linear function of two random variables:*

$$\begin{aligned}\mathbb{E}(W) &= \mathbb{E}(aX + bY) \\ &= a\mathbb{E}(X) + b\mathbb{E}(Y)\end{aligned}$$

Property 1.10 *The variance of a linear function of two random variables:*

$$var(aX + bY) = a^2 var(X) + b^2 var(Y) + 2ab \cdot cov(X, Y)$$

Proof. Start with the definition of variance,

$$\begin{aligned}
\text{var}(aX+bY) &= \mathbb{E}[(aX+bY)-\mathbb{E}(aX+bY)]^2 \\
&= \mathbb{E}[a(X-\mathbb{E}(X))+b(Y-\mathbb{E}(Y))]^2 \\
&= \mathbb{E}[a^2(X-\mathbb{E}(X))^2+b^2(Y-\mathbb{E}(Y))^2+2ab(X-\mathbb{E}(X))(Y-\mathbb{E}(Y))] \\
&= a^2\mathbb{E}(X-\mathbb{E}(X))^2+b^2\mathbb{E}(Y-\mathbb{E}(Y))^2+2ab\cdot\mathbb{E}[(X-\mathbb{E}(X))(Y-\mathbb{E}(Y))] \\
&= a^2\text{var}(X)+b^2\text{var}(Y)+2ab\cdot\text{cov}(X,Y)
\end{aligned}$$

■

Property 1.10 helps us determine the benefits of diversification. Example 1.14 shows how this works in practise with a simple two asset example.

Example 1.14 *Let W be a portfolio of financial assets X and Y, with respective weights $a = 0.5$ and $b = 0.5$,*

$$W = 0.5X + 0.5Y$$

What is the variance of W when $var(X) = var(Y)$ and

(a) $corr(XY) = 1$,

(b) $corr(XY) = 0.5$,

(c) $corr(XY) = 0$,

(d) $corr(XY) = -1$?

Solution 1.14 *(a)*

$$\begin{aligned}
var(W) &= a^2var(X)+b^2var(Y)+2ab\cdot cov(X,Y) \\
&= a^2var(X)+b^2var(Y)+2ab\cdot corr(X,Y)\sigma_X\sigma_Y \\
&= (0.5)^2var(X)+(0.5)^2var(Y)+2(0.5)(0.5)(1)\sigma_X\sigma_Y \\
&= (0.5)^2\sigma_X^2+(0.5)^2\sigma_X^2+2(0.5)(0.5)\sigma_X^2 \\
&= [(0.5)^2+(0.5)^2+2(0.5)(0.5)]\sigma_X^2 \\
&= \sigma_X^2
\end{aligned}$$

(b)

$$\begin{aligned}
var(W) &= (0.5)^2var(X)+(0.5)^2var(Y)+2(0.5)(0.5)(0.5)\sigma_X\sigma_Y \\
&= [(0.5)^2+(0.5)^2+2(0.5)(0.5)(0.5)]\sigma_X^2 \\
&= 0.75\sigma_X^2
\end{aligned}$$

(c)

$$
\begin{aligned}
var(W) &= (0.5)^2 var(X) + (0.5)^2 var(Y) + 2(0.5)(0.5)(0)\sigma_X \sigma_Y \\
&= [(0.5)^2 + (0.5)^2]\sigma_X^2 \\
&= 0.5\sigma_X^2
\end{aligned}
$$

(d)

$$
\begin{aligned}
var(W) &= (0.5)^2 var(X) + (0.5)^2 var(Y) + 2(0.5)(0.5)(-1)\sigma_X \sigma_Y \\
&= [(0.5)^2 + (0.5)^2 - 2(0.5)(0.5)]\sigma_X^2 \\
&= 0
\end{aligned}
$$

Example 1.14 shows how portfolios with more highly correlated assets will have greater variance, all else equal.

Independence

Definition 1.7 *Random variables X and Y are* independent *if and only if*

$$P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j) \qquad \forall\ i, j$$

Notation: We write $X \perp Y$.

Property 1.11 *Random variables X and Y are independent if and only if the marginal probability of the event $X = x_i$ is equal to the conditional probability of the event $X = x_i$ given the realisation of $Y = y_j$,*

$$X \perp Y \text{ iff } P(x_i) = P(x_i | y_j) \quad \forall\ i, j.$$

Property 1.12 *If random variables X and Y are independent, it follows that the covariance of X and Y is zero,*

$$X \perp Y \ \Rightarrow \ cov(X, Y) = 0.$$

Summary and next steps

In Chapter 3, we'll return to discounting using fixed interest rates and arbitrage principles. We'll study how the present value of an asset responds to fluctuations in interest rates. In Chapter 4, we'll use the principle of arbitrage to value forward contracts, a simple form of derivative contract.

In the second half of the course, from Chapter 7 onwards, we'll turn to situations where a replicating portfolio is difficult to construct, and we need to rely on economic principles to discount future risky payoffs.

Problems for Chapter 1

Exercise 1.1 *Prove Property 1.3 for discrete random variables.*

Exercise 1.2 *a. Prove Property 1.7 and*

b. Prove Property 1.8.

Exercise 1.3 *Sam is an investor with* GBP£1 *to invest in a portfolio of assets* X *and* Y*. Sam's portfolio* W*, can be represented by the following expression,* $W = aX + (1-a)Y$*, where* $a \in [0,1]$.

a For given a*, what is the variance of* W*?*

b Find a^* *that minimises the variance of* W*. That is, find*

$$a^* = \arg\max_{a \in [0,1]} var(W) = var(aX + (1-a)Y)$$

c Show that when $var(X) = var(Y)$*, portfolio weight* a^* *does not depend on* $cov(X,Y)$.

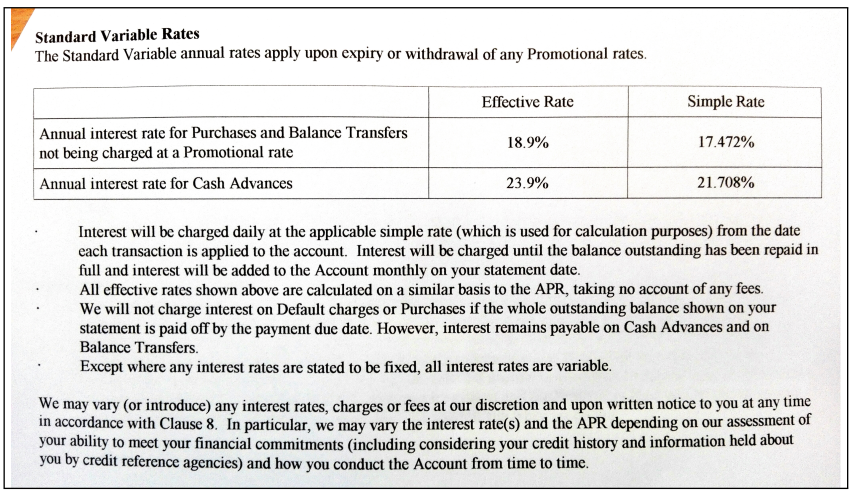
d Under what conditions is (i) $a^* = 1$*, or (ii)* $a^* = 0$*?*

Exercise 1.4 *Consider the random variables* X *and* Y,

		Value of Y		
		-1	0	1
Value of X	1	0.25	0.00	0.25
	-1	0.00	0.50	0.00

Show that X *and* Y *have zero covariance but are not independent.*

Exercise 1.5 *Consider the interest rate information in Figure 1.4, which is taken from documentation provided by a well-known bank. Confirm that the simple and effective interest rates quoted are consistent with monthly compounding.*



Standard Variable Rates
The Standard Variable annual rates apply upon expiry or withdrawal of any Promotional rates.

	Effective Rate	Simple Rate
Annual interest rate for Purchases and Balance Transfers not being charged at a Promotional rate	18.9%	17.472%
Annual interest rate for Cash Advances	23.9%	21.708%

- Interest will be charged daily at the applicable simple rate (which is used for calculation purposes) from the date each transaction is applied to the account. Interest will be charged until the balance outstanding has been repaid in full and interest will be added to the Account monthly on your statement date.
- All effective rates shown above are calculated on a similar basis to the APR, taking no account of any fees.
- We will not charge interest on Default charges or Purchases if the whole outstanding balance shown on your statement is paid off by the payment due date. However, interest remains payable on Cash Advances and on Balance Transfers.
- Except where any interest rates are stated to be fixed, all interest rates are variable.

We may vary (or introduce) any interest rates, charges or fees at our discretion and upon written notice to you at any time in accordance with Clause 8. In particular, we may vary the interest rate(s) and the APR depending on our assessment of your ability to meet your financial commitments (including considering your credit history and information held about you by credit reference agencies) and how you conduct the Account from time to time.

Figure 1.4: Periodic compounding in retail finance.

2

Markets and Instruments

> I coulda bought a place in Dumbo before it was Dumbo
> For like two million
> That same building today is worth twenty-five million
> Guess how I'm feelin'? Dumbo
> ...
> I bought some artwork for one million
> Two years later, that shit worth two million
> Few years later, that shit worth eight million
> I can't wait to give this shit to my children
> Jay Z, *The Story of O.J.*

Introduction and overview

This Chapter introduces the main capital markets and instruments used by firms. Special attention is paid to the distinctions between debt and equity instruments, as well as distinctions between intermediated and decentralised markets.

Capital Markets

We start with a brief overview of the main types of capital markets we'll consider in this course. The first two categories, Equity and Debt markets, will feature heavily throughout the course, and it is these two capital markets that play the primary role of raising funds for firm investment. Commodity and physical asset markets provide opportunities to trade control rights over factors of production, while foreign exchange (FOREX) and derivative markets predominantly serve a risk management purpose (note that risk *management* does not necessarily mean risk *mitigation*).

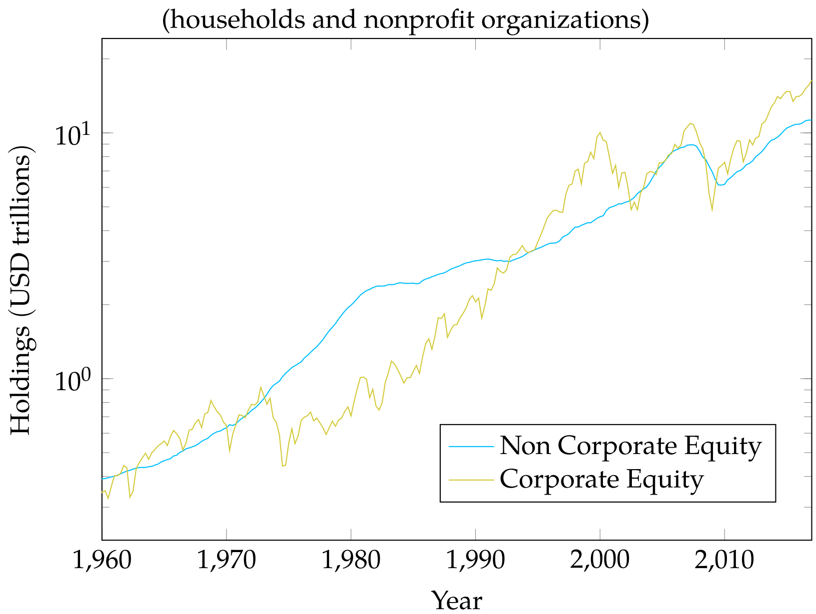
Equity markets

Equity or shares are ownership claims on the firm. Shareholders typically enjoy some control rights including the right to elect directors as well as participate in some major decisions made by the firm. Shareholders also have a right to residual income generated by the firm. As this income is uncertain, the dividend income returned to shareholders and the value of shares themselves are volatile.

Equity markets take many forms. Internal markets for equity can involve the issuing of shares to employees and other firm insiders, as well as the allocation of shares between different firms within a conglomerate. External markets for equity also take numerous forms. Private markets for equity are characterised by long term relationships between individual shareholders and management. Examples include the angel investment popularised by television shows *Dragon's Den* and *Shark Tank*, and also the ownership of most football clubs. Private equity financing is popular at the early stage of firms. Historically, privately placed equity issuance it was limited to small scale fundraising, but more recently it is also used for larger scale fundraising. Uber, for example, has raised USD$8.7 billion in private equity fundraising.[1]. Traditionally, firms have turned to public markets for large-scale equity fundraising. Public markets offer liquidity for shareholders; it is easier to buy and sell holdings publicly traded shares than holdings of privately placed shares. While publicly traded stocks dominate the news and media discussions of the strength of economies, much of the

[1] https://www.crunchbase.com/organization/uber#/entity

equity of firms in most economies is privately placed. 2.1 gives some indication of the relative values of the total markets for private and public equity holdings in the United States.



Source: Federal Reserve Board Z1 tables
(Series identifiers: Z1/Z1/FL152090205.Q, Z1/Z1/FL153064105.Q)

Figure 2.1: Holdings of noncorporate and corporate equity

Debt markets

Debt contracts confer the holder the right to a fixed stream of payments; typically this means a periodic stream of coupons plus a large terminal payment, the principal. Similar to equity, there are private and public markets for debt. Bonds are typically issued in public markets and are tradeable between decentralised bondholders. Private markets for debt include but are not restricted to bank lending markets.

Derivative markets

A *derivative* is a contract that commits parties to payments and actions contingent on verifiable events. A derivative is said to derive value from the values of other assets. It is perhaps easiest to consider some examples of derivatives.

A *forward* contract commits one party to pay/receive the difference between a market price and a pre-determined strike price at some given future date. For example, Apple might estimate that they will sell GBP £10 billion worth of products in the UK next year. In order to mitigate currency risk arising from fluctuations between USD and GBP, Apple might wish to sell forward GBP £10 billion in dollars. This contract would commit Apple to pay the difference between the future GBP/USD exchange rate and the contracted strike exchange rate, if the difference is positive (that is, if GBP appreciates). If negative, Apple would receive the difference from the counterparty.

A *swap* agreement is a bundle of forward contracts. For example, an interest rate swap might commit Alex to pay Bernie the difference between a variable floating interest rate and a pre-determined fixed interest rate when the floating interest rate exceeds the fixed interest rate. When the fixed rate exceeds the floating rate, Bernie would pay Alex the difference. This type of swap contract can be helpful if Bernie had an outstanding bank loan with variable interest payments. The swap, with third party Alex, would protect Bernie from the risk of interest rate increases.

One problem with a forward contract is that they can create large counterparty exposures; large movements in the price of the underlying security require large settlement payments on the termination date. These large settlement payments may be unaffordable to the payer, introducing *counterparty risk*. This counterparty risk may be manageable when the counterparties are familiar to each other. However, this counterparty risk does prevent trade between decentralised, unfamiliar counterparties. A *futures* contract is an exchange traded product that demands continuous maintenance of margin held with a centralised clearing house, largely eliminating counterparty risk. Due to the exchange traded nature of futures contracts and lack of counterparty risk, it is not necessary to match individual buyers and sellers, but rather just to match aggregate demand and supply.

A *European call option* has three inputs. First, an underlying asset is specified (as an example, let's say Apple shares: AAPL). Second, a strike price is specified (say USD$200). Third, a termination date is specified (say 16:00 EST, 31 December 2016). The call option gives the purchaser the right to buy the underlying at the strike price at the termination date. In this example, the purchaser would have the right to buy an Apple share from the seller for $200 at 16:00 EST, 31 December 2016. At the time of writing (20 September 2016) Apple shares are currently trading at USD$113.58. How much would you pay for this option?

Equity and Debt

Debtholders have a senior claim on the firms assets, above that of shareholders. This means that if a firm defaults on their debts, the debtholders have the right to take control of the firm from the shareholders. Figure 2.2 presents a payoff diagram for equity and debt. When the asset value of the firm is below the value of the firm's debts, the equity is worth nothing and the total value of the debt is equal to the value of the firm's assets. When the asset value of the firm exceeds the value of the firm's debst, the value of the firm's equity is the difference between the value of the firm's assets and the value of the firm's debts.

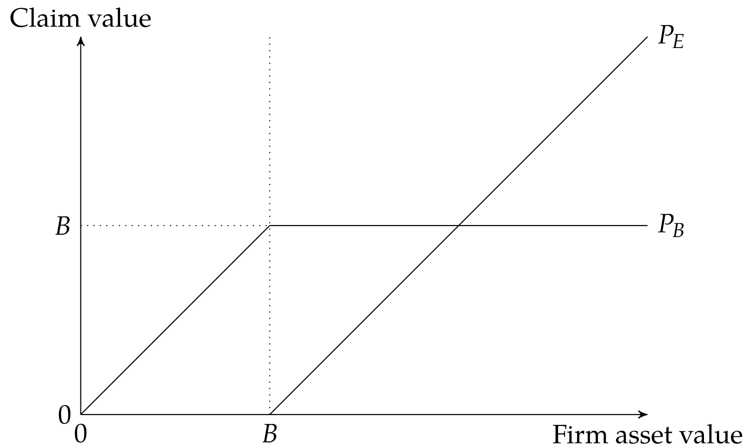


Figure 2.2: Firm asset value and investor payoffs

Why do firms sell equity?

It is useful to stop and ask why there are markets for these instruments in the first place. Why would a firm sell equity and how do they decide between issuance of equity and debt?

Elliot is an entrepreneur with initial wealth w^e and Sam is a saver with initial wealth w^s. Both Elliot and Sam are risk averse, sharing common utility function $u(c)$ where $u', -u'' > 0$. Elliot has access to an investment project returning $y = z \cdot r(a)$, where a is the amount invested in the project, z is a random variable, $z = z_i$ with probability π_i, $r(a)$ is concave, $r, r', -r'' > 0$. All remaining wealth earns gross return 1.

Efficient allocations

What we want is to determine what are the features of an optimal finance contract between Sam and Elliot. An external finance contract is considered *optimal* if the equilibrium outcomes under the contract are superior in expectation to the equilibrium outcomes under any other contract.

We take an approach called *mechanism design*. [2] provides a nice introduction to this approach to the study of financial markets with closely related examples.[3] What we'll do is solve for the consump-

[2] Nobuhiro Kiyotaki. *A Mechanism Design Approach to Financial Frictions*, pages 177--187. Palgrave Macmillan UK, London, 2012. ISBN 978-1-137-03425-0

[3] I also have some research that uses this method .

Alfred Duncan. Private information and business cycle risk sharing. Working Papers 2016-02, Business School - Economics, University of Glasgow, January 2016 tion allocations and actions that maximise a weighted average of the expected utilities of the two agents. Sometimes, this is referred to as the *social planner's problem*. The planner weights (or *Pareto weights*) applied to the utility of each agent can represent market power or initial wealth or other considerations.

We can solve the planner's problem with Pareto weights μ^e, μ^s:

$$\max_{c_i^e, c_i^s, a} \mathbb{E}[\mu^e u(c_i^e) + \mu^s u(c_i^s)]$$

subject to the budget constraints

$$z_i \cdot r(a) + (w^e + w^s - a) \geq c_i^e + c_i^a \qquad \forall\ i.$$

The first term on the left hand side, $z_i \cdot r(a)$, is the (stochastic) return to Elliot's investment project. The second term, $w^e + w^s - a$, is the remaining wealth of Elliot and Sam that has not been allocated to the project. The right hand side, $c_i^e + c_i^a$, is the sum of the consumption of Elliot and Sam. The notation $\forall\ i$ means *for all states i*.

We solve this problem using the Lagrangian method. The Lagrangian for this problem can be expressed as follows:

$$\mathcal{L} = \mathbb{E}[\mu^e u(c_i^e) + \mu^s u(c_i^s) + \lambda_i(z_i \cdot r(a) + (w^e + w^s - a) - c_i^e - c_i^a)]$$

At an optimum, the derivatives of the Lagrangian $\mathcal{L}$ with respect to the choice variables c_i^j, a and the Lagrange multipliers λ_i are all equal to zero. These conditions are referred to as the *first order necessary conditions* of the Lagrangian.

In this example, the first order necessary conditions are

$$\frac{\partial \mathcal{L}}{\partial c_i^j}: \qquad 0 = \pi_i[\mu^j u'(c_i^j) - \lambda_i], \tag{2.1}$$

$$\frac{\partial \mathcal{L}}{\partial a}: \qquad 0 = \mathbb{E}[\lambda_i(z_i \cdot r'(a) - 1)]. \tag{2.2}$$

For our purposes, the important condition is 2.1, which we can rewrite as:

$$\mu^e u'(c_i^e) = \mu^s u'(c_i^s) \qquad \forall\ i$$

$$\frac{u'(c_i^e)}{u'(c_j^e)} = \frac{u'(c_i^s)}{u'(c_j^s)} \qquad \forall\ i, j \tag{2.3}$$

Equation 2.3 tells us that Elliot and Sam share the risk of production. The left (right) hand side of 2.3 is Elliot's (Sam's) marginal rate of substitution from consumption in state j to consumption in state i.

You may be unfamiliar with thinking about marginal rates of substitutions between consumption in two different states of the world, but they work the same way as standard marginal rates of substitution in other areas in economics. The marginal rate of substitution $\frac{u'(c_i^e)}{u'(c_j^e)}$ is the amount of consumption in state j that Elliot would give up to increase their consumption in state i by one unit.

We can go somewhat further by imposing restrictions on the preferences of Elliot and Sam. Let $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, then

$$\frac{c_i^e}{c_j^e} = \frac{c_i^s}{c_j^s} \qquad \forall\ i, j. \tag{2.4}$$

Equation 2.4 states that the consumption allocations of Elliot and Sam move together. When Elliot has high consumption, Sam has high consumption. The optimal contract is therefore a risk sharing contract, where Sam as the outside investor enjoys higher consumption when the payoff from the investment is high. In other words, the optimal contract involves Sam holding shares in the project.

Lesson: Shares help entrepreneurs and savers share the investment and production risks.

Why do firms sell debt?

If equity helps firms and savers effectively share productive risk, then why would firms issue debt contracts? Debt contracts commit firms to constant repayments, regardless of income. This concentrates productive risk with borrowers.

Starting from the same model, let's assume that the state i is revealed to Elliot only. If the true revenue is y_i, Elliot can declare revenue y_k, and hide the difference $y_i - y_k$. We can solve this problem by appealing to the *revelation principle*, a powerful tool for the study of economic problems involving information asymmetries.

The revelation principle: Any optimal allocation can be achieved through a direct mechanism that encourages truthful reporting by all agents.

Applying the revelation principle, we impose the restraint on contractual arrangements that borrowers must have an incentive to accurately report their income in all states of the world. This truth-telling constraint is:

$$c_k^e + y_i - y_k \leq c_i^e \qquad \forall\ i, k$$

The only way to satisfy this truth-telling constraint is to ensure that

$$c_k^e - c_i^e = y_k - y_i \qquad \forall\ i, k$$

Elliot retains all productive risk. If Sam contributes to the funding of the project, it must be through a `debt' contract with a fixed repayment.

Sometimes, this property of debt is referred to as *information insensitivity*. The borrower Elliot cannot exploit their information advantages over Sam; repayments are not contingent on reported income.

Lesson: Debt helps entrepreneurs and savers manage moral hazard.

Markets, intermediaries and conglomerates

The previous section considered the firm's choice between issuing debt and issuing equity. But as we saw earlier, each of these types of instruments can be issued in more centralised, private markets or more decentralised, public markets. What determines the choice between issuing publicly tradable liabilities or non-tradable liabilities?

Type of Market			
	Public Markets	Intermediaries	Internal Financing
Instruments and Risk Sharing Characteristics			
Equity	Public Equity	Private Equity	Conglomerates
Debt	Bonds	Bank lending	
Further Characteristics			
	Decentralised	$\longrightarrow$	Centralised
	Less monitoring	$\longrightarrow$	More monitoring
	Less flexibility	$\longrightarrow$	More flexibility
	More liquidity	$\longrightarrow$	Less liquidity

Table 2.1: Financial arrangements, markets and characteristics

Table 2.1 presents a break down of some of the characteristics of different financing arrangements. Publicly traded liabilities have more decentralised ownership. This increases liquidity of these instruments, which can increase demand from investors. On the other hand, having a more dispersed, decentralised set of shareholders and creditors can make it difficult to develop relationships between the firm and its shareholders and creditors. This might make it difficult for shareholders and creditors to monitor firm management and ensure that the firm is being run effectively. In addition, it may make it difficult to renegotiate the terms of these contracts in the event that the firm's income deteriorates. More centralised, private and internal placements of equity and debt help foster long term relationships between the firm and its investors. This helps encourage effective monitoring, and allows for the opportunity of renegotiation when appropriate.

Leverage patterns by industry

Plot 2.3 presents the relationship between firm assets and leverage for Financial and Energy sector firms in the S&P 500. On average, financial sector firms have higher leverage than energy sector firms---this should be unsurprising. Somewhat more interesting is the difference in the relationship between firm size and leverage across the two sectors. Energy sector firms appear to fund expansion with proportionate increases in equity and debt; energy sector leverage is approximately invariant to firm size. Financial sector firms appear to fund expansion with a disproportionate reliance on debt finance relative to equity finance.

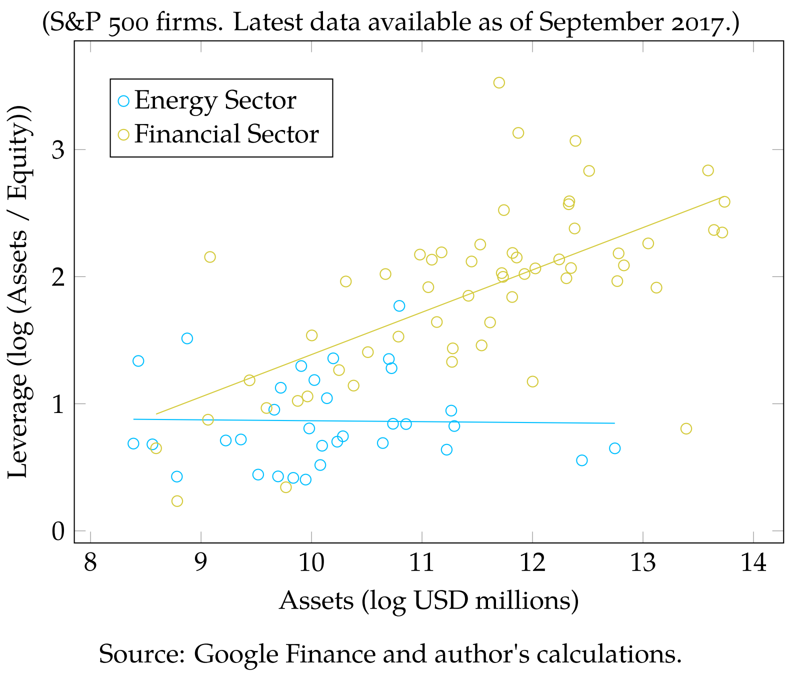


Figure 2.3: Assets and Leverage in the Financial and Energy Sectors

Stock markets around the world

Table 2.2 presents stock market capitalisation to GDP measures for a selection of countries. What is perhaps surprising is the wide variation in stock market capitalisation across seemingly similar countries. What are the determinants of market capitalisation across countries?

Legal systems Well functioning public stock markets require strong legal systems that can bring together decentralised investors in dispute with firms. In the context of the United States, the Delaware Court of

Chancery leads the world in the settlement of corporate disputes, a fact that is often credited for the size of public stock markets in the US. However, while the UK's strong legal institutions may be one factor in determining the fact that the UK has larger stock markets than Italy, but it doesn't really help us when we compare the UK with South Africa, or Honduras.

Income As a country grows in income, savings increase and the capital stock increases. This investment requires the mobilisation of savings, perhaps through stock markets. But the data suggest that there are not necessarily strong links between income and stock market capitalisation. There are rich countries with small stock markets (Italy) and low income countries with large stock markets (India, South Africa).

Taxes We might expect that in countries with lower corporate tax rates, there would be greater use of equity finance and consequently greater public stock markets relative to GDP. Across European countries, this seems to have some explanatory power. Switzerland for example has low corporate income tax rates and high stock market capitalisation. But other low corporate tax countries including the Cayman Islands and Bermuda don't follow this trend.

Gateways Three countries that jump out of this data are Hong Kong, Switzerland and South Africa. These countries' stock markets appear to provide gateways for international investors into a wider set of countries. For example, firms from China issue shares in Hong Kong, which can be bought by international investors.

Case study: home loans or home shares?

Most home purchases are funded through debt contracts called mortgages. These mortgages leave the homeowner exposed to fluctuations in the value of the property, some of which are out of their control. Recessions, natural disasters and changes in school zones affect property values and can result in large fluctuations in the wealth of the homeowner. In the wake of the global financial crisis, falls in housing worth relative to mortgage liabilities reduced household spending, increasing the severity of the recession [4]. Equity contracts, as an alternative to mortgages, would share these risks between the homeowner and the more lender, potentially increasing welfare.

[4] Atif Mian and Amir Sufi. Household leverage and the recession of 2007-09. *IMF Economic Review*, 58(1):74--117, 2010

Table 2.2: Stock market capitalisation for selected countries

(Source: World Bank, latest available as at August 2016)

Country	Stock Market Capitalisation (% of GDP)	GDP per capita (USD)	GDP (USD b)
Asia and the Pacific			
Australia	89	56 328	1 340
China	75	7 925	10 866
Hong Kong	1 028	42 423	310
India	73	1 582	2 074
Japan	119	32 477	4 123
Malaysia	129	9 766	296
Singapore	219	52 889	293
Europe			
Channel Islands	15	73 568	12
France	86	36 248	2 422
Germany	51	41 219	3 356
Italy	27	29 847	1 815
Luxembourg	82	101 450	58
Norway	50	74 735	388
Switzerland	229	80 215	665
United Kingdom	67	43 734	2 849
Latin America and the Caribbean			
Barbados	105	15 661	4
Brazil	28	8 539	1 775
Cayman Islands	5	6 4105	3
Honduras	538	2 496	20
Middle East			
Kuwait	105	28 985	113
Qatar	85	74 667	167
North America			
Bermuda	26	85 748	6
Canada	103	43 249	1 551
United States	140	55 837	17 947
Africa			
South Africa	235	5 692	313

Shared responsibility loans

[5] argue that it would be better if residential finance took the form of home *shares*, rather than home loans. Under their proposal, lenders and borrowers would share any appreciation or depreciation in home prices. Risk sharing real estate finance is rare but does exist in a range of forms. In Canterbury, there are typically property listings that offer

[5] Atif Mian and Amir Sufi. *House of Debt*. University of Chicago Press, 1 edition, 2015

shared ownership to the purchaser---essentially the purchaser and the seller each hold equity shares in the property.[6] PartnerOwn is a startup offering shared responsibility loans in the US.[7] PartnerOwn's Shared Responsibility mortgages are linked to the value of local market prices, with loan repayments falling if local market prices fall (**?**, describes a similar model).

[6] See https://www.rightmove.co.uk/property-for-sale/Canterbury/shared-ownership.html.
[7] See http://partnerown.com/.

Ninety-year-old widow faces sickening £177k bill to repay a £22k mortgage - but Barclays insists its 690% profit is fair

- **The £177k bill is triggered when Mrs T sells her home**
- **Mrs T cannot afford to downsize because she wouldn't be left with enough money once the mortgage had been paid**
- **Mortgages sold in the 1980s have since been banned**

By SARAH DAVIDSON FOR THISISMONEY.CO.UK
PUBLISHED: 08:02, 2 August 2016 | **UPDATED:** 12:23, 2 August 2016

Figure 2.4: Risk sharing mortgages in the media

Source:
http://www.dailymail.co.uk/money/mortgageshome/article-3684914/Barclays-charged-90-year-old-widow-177-750-repay-22-250-shared-appreciation-mortgage.html

There are two important issues or concerns that emerge when home finance contracts involve more risk sharing. The first concern, if repayments are tied to the value of the property, there may be underinvestment in home improvements. Appreciation due to improvements is shared between owner and lender, why would the homeowner add improvements to the house if any resulting house price increase is shared with the lender? Similarly, some of the costs of depreciation resulting from poor maintenance would be passed on to lenders, stoking moral hazard. One response to this is to tie mortgage value to wider house price indices, rather than the value of the particular house (this is the approach taken by PartnerOwn). But this is technically challenging in many cases.

The second concern is that these contracts would not always reduce consumption risk of households. A standard debt-financed mortgage places all of the appreciation risk with the borrower. This means that if house prices increase, the borrower can always afford to move into a similar house, perhaps in order to take a new job. Shared responsibility mortgages, by restricting the borrower's exposure to appreciation risk, can prevent borrowers from being able to purchase similar properties if they need to move. This problem is captured in the *Daily*

Mail story referred to in Figure 2.4. In this case, the pensioner homeowner could not afford to *downsize* as a result of the large increases in property values since the initiation of the shared responsibility-type mortgage.

Aside from these concerns with the risk sharing aspects of shared responsibility mortgage contracts, a further question is to ask, how house price risk *should* be allocated between households. While the introduction of Shared Responsibility Mortgages would shift house price risk away from indebted households, it would also shift this risk toward other agents in the economy, including banks. It is not clear that there is much appetite to increase the exposure of savers and the financial sector to house price fluctuations.

Case study: Student loans or student shares?

Let's consider the main features of student loans in the United Kingdom. Interest is charged at a reasonably high rate (Retail Price Index inflation plus 3%). Repayments are made only when gross income exceeds a threshold (currently the threshold is £21,000). The minimum repayment rate is 9% of the difference between gross income and the income threshold when positive. When the principal is fully paid, repayments cease. 30 years after repayments start, the balance of the debt is written off and repayments cease.

We can summarise these features as follows: Low income borrowers pay nothing (like equity). Middle income borrowers repayments tied to income (like equity). Repayments of high income borrowers capped by loan value (like debt).

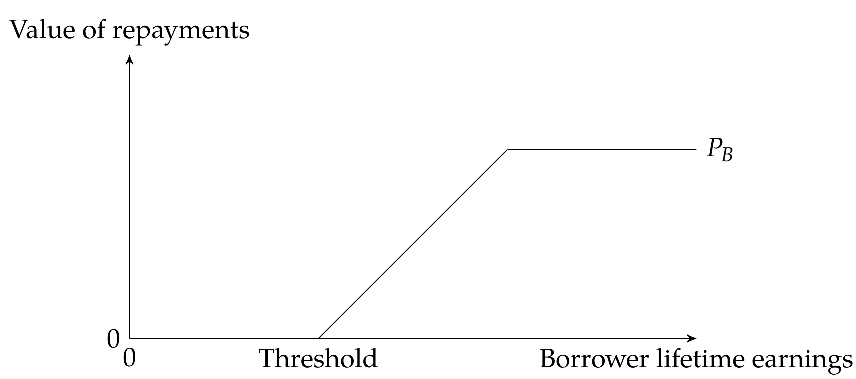


Figure 2.5: Borrower earnings and the value of student loan repayments

The payoff to the taxpayer for a given student loan is loosely approximated by Figure 2.5. Repayments are contingent on income when income is sufficiently but not too high. This does allow a high degree of risk sharing. It also means that on average, philosophy majors will repay less than financial economics majors.

Problems for Chapter 2

Exercise 2.1 *What is the difference between a* forward *contract and a* futures *contract? In what situations would you expect a firm to favour the use of forward contracts over futures contracts and vice-versa?*

Exercise 2.2 *Technology firms typically raise external funds by issuing equity, rather than by issuing debt. Utilities (for example electricity generating firms) typically raise external funds by issuing debt, rather than by issuing equity. Discuss this distinction in the context of the motivations for equity and debt finance considered in this Chapter.*

Exercise 2.3 *Many technology firms initially raise funds through private equity, then later through public stock markets. What explains the initial preference for private funding sources and the later preference for public funding sources?*

Exercise 2.4 *With notable exceptions (including Manchester United, Celtic Football Club, Ajax, Sporting Lisbon, Porto, Roma, Juventus and Lazio) football clubs typically do not sell shares in public markets. Rather, shares in football clubs are typically traded in private markets. What do you think are the main considerations determining the choice between the public or private financing of football clubs?*

To help answer this question, or just for your own interest, you may consider reading Manchester United's latest Annual Report:
`http://ir.manutd.com/~/media/Files/M/Manutd-IR/Annual%20Reports/2015-20f.pdf`

Exercise 2.5 *Small firms typically raise debt finance through bank loans. Larger firms are more likely to raise debt finance through bond markets. Why?*

Exercise 2.6 *In the United States, firms are more likely to raise funds in decentralised markets (that is, by issuing publicly traded shares and bonds) than in Europe, where firms rely more on bank loans. Why?*

Exercise 2.7 *Do you think that the UK student loan scheme offers the appropriate balance between risk sharing and moral hazard? Explain.*

Exercise 2.8 *Do you think that a Graduate Tax, that is an additional flat-rate income tax for former university students, would be a better way to fund university tuition?*

Exercise 2.9 *Do you think that standard mortgage contracts offer the appropriate balance between risk sharing and moral hazard? Explain.*

Exercise 2.10 *Hong Kong has the world's largest ratio of stock market capitalisation to GDP. Why? (Your answer should consider multiple competing* *hypotheses, as well as global patterns in stock market capitalisation to GDP ratios.)*

3
Fixed income

Be sure you know the condition of your flocks,
give careful attention to your herds;
Proverbs 27:23

You wanna know what's more important than throwin' away money at a strip club? Credit
Jay Z, *The Story of O.J.*

Introduction

In this chapter we develop tools to understand the relationship between the value of a bond and market interest rates. When market interest rates increase, money set aside in an account will increase in value more quickly, all else equal. This means we need less funds today to replicate an asset with fixed future payoffs. The upshot is that the value of a bond is inversely related to market interest rates.

In this chapter, we develop formulas to determine the present values of standard types of bonds, the sensitivities of these present values to fluctuations in market interest rates (Duration). We then consider the sensitivity of Duration itself to fluctuations in market interest rates (Convexity) and discuss the importance of bond Convexity for valuation and risk management.

As in Chapter 1, we discount future payments into present values by considering how much money we would need today to purchase a portfolio of assets that would replicate the future payments. This approach gives us a correspondence between market interest rates available on savings assets, and the *discount rates* we use to value future payments.

In this chapter, we will focus on assets that provide a risk free sequence of payments. This contrasts with future chapters that pay more attention to risk across states but do not emphasise the dynamic sequence of payments. Also, in this chapter, we will assume that interest rates are constant across maturities at any point in time. We will think of a fall in the discount rate as being a fall in the interest rate at each maturity. The reason for this assumption is of course to simplify the math and intuition. We had better quickly look at some data to get an idea of how unrealistic this assumption is.

Figure 3.1 plots the historical behaviour of annual effective interest rates for 1 year and 10 year US Government bonds. The assumption in this chapter is that these interest rates are identical at each point in time. It is clear from Figure 3.1 that this assumption is going to result in some error. Over the past 50 years, these interest rates have tended to move together over long periods, but at shorter frequencies these long and short term interest rates can diverge. We need to keep this weakness in mind when we work with the material in this chapter.

Present value

Recall from Chapter 1 that under continuous compounding, the future value at time t, $A(t)$ of an account with initial amount invested A_0 is given by

$$A(t) = \exp(rt)A_0. \tag{3.1}$$

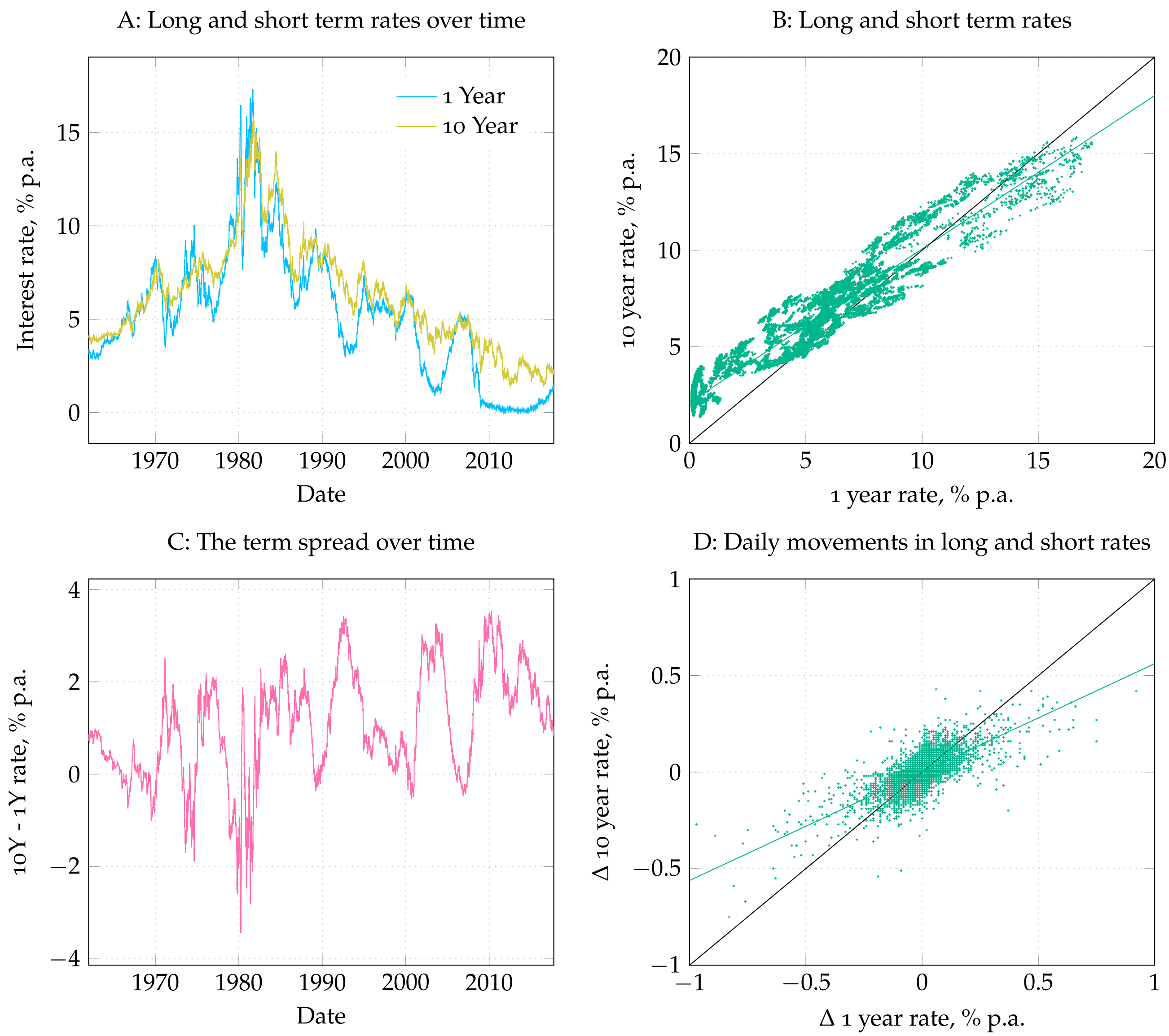


Figure 3.1: Interest rates, long and short (US Government bonds) Source: Fred Database, St Louis Federal Reserve (Series identifiers: DGS1, DGS10)

We say that A_0, the amount invested today, is the present value of $A(t)$. Given future payoff $A(t)$ along with the time of the payoff t and the interest rate r, we can derive the present value v from Equation 3.1:

$$\begin{aligned}
\exp(rt)v &= A(t) \\
v &= \frac{1}{\exp(rt)} A(t) \\
v &= \exp(-rt)A(t). \qquad (3.2)
\end{aligned}$$

Throughout this chapter, we will be applying this formula to sequences of future payments.

A bond b is defined by n cash flows x_i that occur at dates t_i, for $i \in \{1, 2, 3, ..., n\}$. Let r be the continuously compounding discount rate. The discount rate r is typically determined by the market interest rate, which can be thought of as the opportunity cost of funds. For the purposes of this lecture, we abstract from risk premia and from the yield curve, assuming a constant discount rate for all future payments. The main lessons from this lecture are still valid after adding in more structure, but the math can get a lot more complicated.

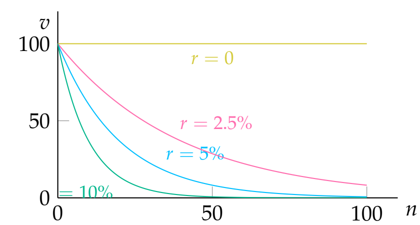
By Equation 3.2, we know that the present value of a future cash flow x is

$$v(x) = \exp(-rt)x,$$

where t is the time in years before the cash flow x is realised.

Figure 3.2 presents the present values (v) of a fixed payment of 100, varying the continuously compounded discount rate (r) and the time to maturity (n). Inuitively, when the discount rate is positive, payments that have a longer time to maturity have a lower present value (the schedules are downward sloping). For all payments with positive time to maturity, the present value of the payment is decreasing in the discount rate (the schedules move to the bottom left corner as discount rates increase).

Figure 3.2: Maturity and present value of a payment of 100 at time n.



A bond, a collection of payments, will also have a present value that is decreasing in the discount rate, and at least in some loose sense is decreasing in the maturity of the payments. For a portfolio manager, understanding the sensitivity of instrument valuations to movements in interest rates is crucial for risk management.

The present value of bond b is denoted by v. To find the present value of a bond, we add up the present values of all the payments:

$$\begin{aligned} v &= \exp(-rt_1)x_1 + \exp(-rt_2)x_2 + \exp(-rt_3)x_3 + ... + \exp(-rt_n)x_n \\ &= \sum_{i=1}^{n} \exp(-rt_i)x_i \end{aligned}$$

Lets look at a few examples. We start with an annuity, which is a schedule of fixed constant payments. Fixed rate mortgages are typically structured as annuities, with constant repayments over the life of the loan.

Example 3.1 *Let b be an annuity, with constant coupon payments x at times $t_i = \{1, 2, 3, ..., n\}$. What is the present value of bond b?*

Trick. Recall that for geometric sums with $0 < z < 1$, we have

$$\sum_{j=1}^{n} z^j = \frac{z(1 - z^n)}{1 - z}.$$

We'll use variations of this trick throughout this chapter, with $z := e^{-r}$ and $j := t$.

Solution 3.1

$$
\begin{aligned}
v &= \sum_{i=1}^{n} \exp(-rt_i)x_i \\
&= \sum_{t=1}^{n} e^{-rt}x \\
&= \frac{e^{-r}(1-e^{-rn})x}{1-e^{-r}} \qquad\qquad = \frac{1-e^{-rn}}{e^{r}-1}x.
\end{aligned}
$$

What happens as the maturity of the annuity increases to infinity? An annuity that never matures is called a *perpetuity*. The UK Government once issued perpetuities, referred to as Consols. Unfortunately for finance nerds, the UK Government has now repurchased all outstanding Consols, with the last of these bonds taken out of circulation in 2015. Perpetuities promise an infinite schedule of repayments. What is their present value?

Example 3.2 *Let b be a perpetuity, with constant coupon payments x at times $t_i = \{1, 2, 3, ..., \infty\}$. What is the present value of bond b?*

Solution 3.2

$$
\begin{aligned}
v &= \sum_{t=1}^{\infty} e^{-rt}x \\
&= \frac{e^{-r}x}{1-e^{-r}} \qquad\qquad = \frac{x}{e^{r}-1}. \\
&\approx \frac{x}{r} \qquad \textit{When } r \textit{ is small.}
\end{aligned}
$$

Example 3.2 shows that the present value of a perpetuity is finite, with a reasonably tractable solution. Figures 3.3 and 3.4 plot the present values of perpetuities against those of annuities with ever increasing maturities. If the maturity of the annuity is sufficiently long, and the discount rate is sufficiently high, then the present value of the annuity will approach that of a perpetuity with the same coupon payments.

Figure 3.3: Maturity (n) and present value (v) of annuities and a perpetuity with equal coupon payments ($x = 5, r = 0.05$).

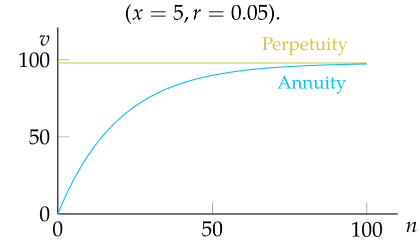
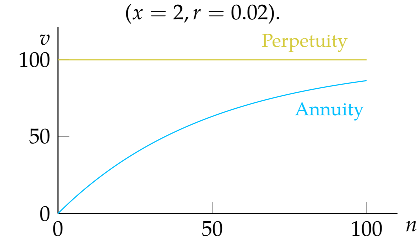


Figure 3.4: Maturity (n) and present value (v) of annuities and a perpetuity with equal coupon payments ($x = 2, r = 0.02$).



Now lets look at a more standard bond structure. A *bullet loan* specifies a series of fixed coupon payments over the life of the loan. The final payment includes both a coupon payment and also the principal payment, which is typically the *face value* of the loan. Most corporate bonds and government bonds have this structure.

Example 3.3 *Let b be a bond with coupon payments $x_i = x$ for $i \in \{1, ..., n-1\}$, and $x_n = x + f$, f is the face value of the bond. Payments* *occur at times $t_i = i$. In words, there is a constant payment x at the end of each year for n years, plus an additional payment of the face value f at the final payment date. Essentially, the bond is a combination of an n period annuity and a zero coupon bond maturing at time n. What is the present value of b?*

Solution 3.3

$$\begin{aligned} v &= \sum_{i=1}^{n} \exp(-rt_i)x_i \\ &= \sum_{t=1}^{n} e^{-rt}x + e^{-rn}f \end{aligned}$$

The first term on the right hand side is just an annuity, so we can use the solution from example 3.1

$$v = \frac{1 - e^{-rn}}{e^r - 1}x + e^{-rn}f.$$

Proposition 3.1 *When the annualised effective discount rate is equal to the coupon rate of a bullet loan ($e^r - 1 = x/f$), the present value is independent of the maturity of the loan.*

Proof. First, we'll take the solution to Example 3.3 and isolate the terms involving maturity, n,

$$\begin{aligned} v &= \frac{1 - e^{-rn}}{e^r - 1}x + e^{-rn}f. \\ &= \frac{x}{e^r - 1} - \frac{e^{-rn}}{e^r - 1}x + e^{-rn}f. \\ &= \frac{x}{e^r - 1} - e^{-rn}f\left[\frac{1}{e^r - 1}\frac{x}{f} - 1\right]. \end{aligned}$$

This expression is insensitive to n when the term in square brackets is equal zero:

$$\begin{aligned} \frac{1}{e^r - 1}\frac{x}{f} - 1 &= 0 \\ \frac{x}{f} &= e^r - 1. \end{aligned}$$

■

Figure 3.5 uses the solution to Example 3.3 to plot the present value of a bond with coupon of 5 and a face value of 100, for a range of maturities and discount rates. When the discount rate is zero ($r = 0\%$), an increase in the maturity of the bond means an increase in coupon payments. These coupon payments are not discounted, so the present value of the bond increases linearly in maturity.

For effective annual discount rates between 0% and 5% (5% is the *coupon rate* in this example), the present value of the bond is increasing in the maturity of the bond. Increases in maturity delay the large face value payment of 100, but they increase the number of coupons received, more than compensating the holder. When the effective annual discount rate is equal to the coupon rate of 5% in this example, The present value of the bond is independent of the maturity, as shown by Proposition 3.1.

For high discount rates, the sensitivity of present values to maturity is low when the maturity is already long. In Figure 3.5, the green schedule is very flat when n is large. Further increases in maturity add payments which are heavily discounted, and they delay the large face value payment which has already been heavily discounted.

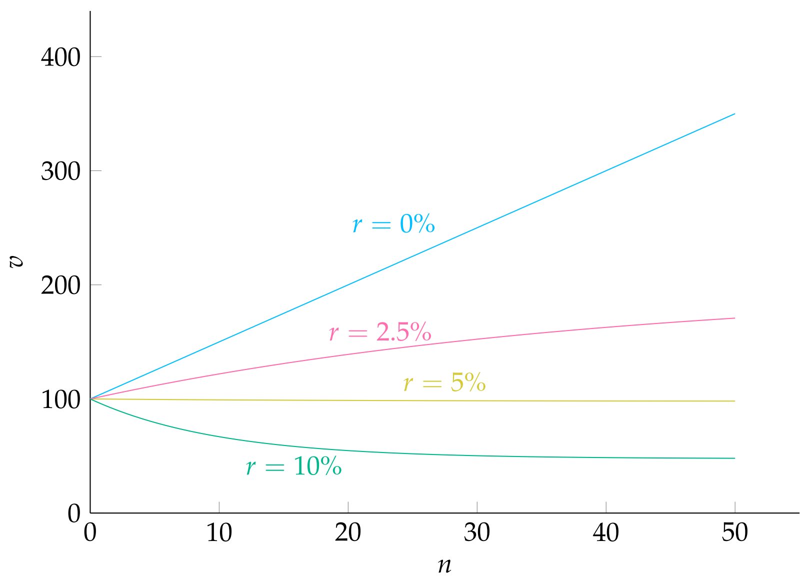


Figure 3.5: Maturity and present value of a bond with annual coupon of 5 and face value of 100, maturing at time n.

Consider two bonds, one with maturity of 30 years and one with maturity of 50 years. From Figure 3.5, we can see that the downside risk for these bonds, the risk of increased interest rates, is similar in magnitude. On the upside however, the sensitivity of prices to interest rates is much greater for the 50 year bond than for the 30 year bond. Figure 3.6 plots the present values for three bonds, maturing at dates 10, 30 and 50. Each bond has a coupon of 5 and a face value of 100. The plot shows how the present values of these bonds change with respect to changes in the discount rate. When interest rates are low, the slope of the schedules is much steeper for the longer maturity bonds than for the shorter maturity bonds. This relationship between maturity and the sensitivity of present values to discount rates is formalised by the notion of *duration*, which we define and explore in the next section (Section 3).

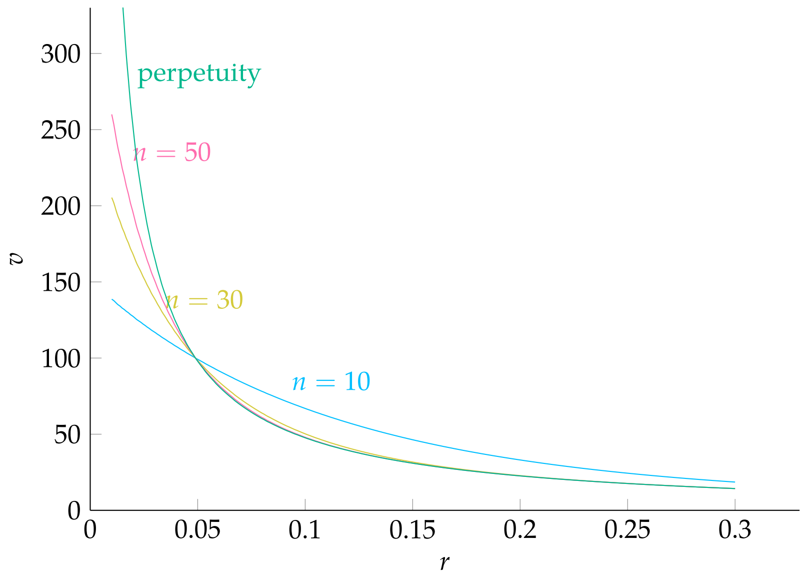


Figure 3.6: Discount rates and present values of bonds with annual coupon of 5 and face value of 100, maturing at 10, 30 and 50 years, and a perpetuity.

Comparing just the 30 and 50 year bonds, the 50 year bond is much more sensitive to discount rate movements when discount rates are low, but behaves similarly to the 30 year bond when discount rates are at or above the coupon rate. Compared with the 30 year bond, the 50 year bond appears to have a lot of upside risk if discount rates fall, with similar downside risk if interest rates increase. This property is formalised by the concept of *convexity*, which we explore in Section 3.

Duration

Duration is a measure of the sensitivity of the present value of a bond to fluctuations in the discount rate. In Figure 3.2 we saw that for short maturity payments, the present value of short maturity payments is insensitive to fluctuations in discount rates; the present value of long maturity payments is very sensitive to fluctuations in discount rates.

For a zero coupon bond, perhaps this link between maturity and sensitivity to discount rate movements is straightforward. When we have a security or bond with many payments, *duration* is a measure that links the maturities of the schedule of payments to the sensitivity of present values to interest rate movements.

Definition 3.1 *The* Modified Duration (duration *for the purposes of this course) of a bond is the semi-elasticity of the present value of the bond, v, to fluctuations in the discount rate r.*

$$D = -\frac{1}{v}\frac{dv}{dr}.$$

Note that this can also be written as $D = -\frac{d \log v}{dr}$.

Proposition 3.2 *Let r be the continuously compounded discount rate. The duration of a bond can be expressed as follows:*

$$D = \frac{\sum_{i=1}^{n} t_i \exp(-rt_i)x_i}{\sum_{i=1}^{n} \exp(-rt_i)x_i}$$

Proof.

$$\begin{aligned}
\frac{dv}{dr} &= \frac{d}{dr}\left[\sum_{i=1}^{n} \exp(-rt_i)x_i\right] \\
&= \frac{d}{dr}\left[\exp(-rt_1)x_1 + \exp(-rt_2)x_2 + \exp(-rt_3)x_3 + ... + \exp(-rt_n)x_n\right] \\
&= -t_1 \exp(-rt_1)x_1 - t_2 \exp(-rt_2)x_2 - t_3 \exp(-rt_3)x_3 - ... - t_n \exp(-rt_n)x_n \\
&= -\sum_{i=1}^{n} t_i \exp(-rt_i)x_i
\end{aligned}$$

$$\begin{aligned}
D &= -\frac{1}{v}\frac{dv}{dr} \\
&= \frac{\sum_{i=1}^{n} t_i \exp(-rt_i)x_i}{\sum_{i=1}^{n} \exp(-rt_i)x_i}
\end{aligned}$$

■

This is ugly. And it is even worse if one does not assume continuously compounding discount rates. But some examples are not too bad.

Note that duration is the present value weighted average of the timing of payments:[1]

$$D = \frac{\exp(-rt_1)x_1}{\sum_{i=1}^{n} \exp(-rt_i)x_i} t_1 + \frac{\exp(-rt_2)x_2}{\sum_{i=1}^{n} \exp(-rt_i)x_i} t_2 + ... + \frac{\exp(-rt_n)x_n}{\sum_{i=1}^{n} \exp(-rt_i)x_i} t_n$$

[1] This only holds when discounting by a continuously compounded discount rate.

Example 3.4 *Let b be a zero coupon bond, with only one payment x occurring at time $t = n$. What is the duration of bond b?*

Solution 3.4

$$\begin{aligned}
D &= \frac{\sum_{i=1}^{n} t_i \exp(-rt_i)x_i}{\sum_{i=1}^{n} \exp(-rt_i)x_i} \\
&= \frac{ne^{-rn}x}{e^{-rn}x} \\
&= n.
\end{aligned}$$

That worked out nicely! For a zero coupon bond, the semi-elasticity of present value to discount rate movements is equal to the maturity of the bond.

Example 3.5 *Let b be an perpetuity, with equal payments $x_i = x$ for $i \in \{1, 2, 3...\}$, occuring at time $t_i = i$. In words, there is a constant payment x at the end of each year. The British government once issed perpetuities as their main source of funding. These bonds were called Consols. What is the duration of b?*

Solution 3.5 *If you already have the formula for duration of an annuity as in Example 3.5, then the easy way is just to take the limit as n approaches infinity. If you don't already have the duration formula for an annuity, solving for the duration of a perpetuity is reasonably straightforward.*

$$\begin{aligned} D &= \frac{\sum_{i=1}^{\infty} t_i \exp(-rt_i)x_i}{\sum_{i=1}^{\infty} \exp(-rt_i)x_i} \\ &= \frac{\sum_{t=1}^{\infty} te^{-rt}x}{\sum_{t=1}^{\infty} e^{-rt}x} \\ D &= \frac{\sum_{t=1}^{\infty} te^{-rt}}{\sum_{t=1}^{\infty} e^{-rt}} \end{aligned} \tag{3.3}$$

By Property 3.4, the denominator of the right hand side of (3.3) is

$$\sum_{t=1}^{\infty} e^{-rt} = \frac{e^{-r}}{1 - e^{-r}}.$$

By Property 3.5, the numerator is

$$\sum_{t=1}^{\infty} te^{-rt} = \frac{e^{-r}}{(1 - e^{-r})^2}.$$

So, we now have

$$\begin{aligned} D &= \frac{\frac{e^{-r}}{(1-e^{-r})^2}}{\frac{e^{-r}}{1-e^{-r}}} \\ &= \frac{1}{1 - e^{-r}}. \end{aligned}$$

Figure 3.7 presents the relationship between bond maturity and duration for a zero coupon bond, an annuity, and two bonds with coupon rates of 1% and 4% respectively. When the maturity of a bond increases, the bond's duration increases. For a zero coupon bond, this relationship is one-to-one. For a coupon bond, this relationship flattens as the maturity of the bond increases. This effect is clear to see even when the coupon is very small (the 1% coupon bond behaves very differently than the zero coupon bond).

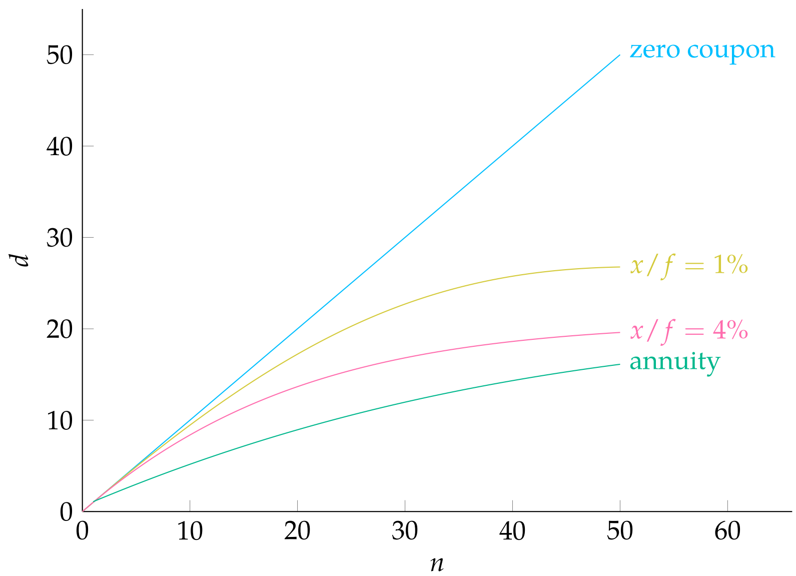


Figure 3.7: Maturity and duration of bonds maturing at time n. The discount rate is $r = 5\%$.

Convexity

Convexity is a measure of the sensitivity of a bond's duration to the discount rate. In Figure 3.6 we compared a perpetuity with some long dated coupon bonds or bullet loans. We saw that the perpetuity had similar duration and present value for discount rates close to the coupon rate. However, when discount rates fell, the perpetuity increased in value more sharply than the 30 and 50 year bonds. In sum, the perpetuity had similar downside risk, but more upside risk. This concept is formalised by the notion of convexity.

Definition 3.2 *The* convexity, C, *of a bond b with present value v, is defined as follows:*

$$C = \frac{1}{v}\frac{d^2v}{dr^2}.$$

Proposition 3.3 *Convexity:*

$$C = D^2 - \frac{dD}{dr}$$

Proof. Starting with the definition of convexity:

$$\begin{aligned}
C &= \frac{1}{v}\frac{d^2v}{dr^2} \\
&= \frac{1}{v}\frac{d}{dr}\left[\frac{dv}{dr}\right] \\
&= \frac{1}{v}\frac{d}{dr}[-vD] \\
&= \frac{1}{v}\left[-D\frac{dv}{dr} - v\frac{dD}{dr}\right] \\
&= -D\frac{1}{v}\frac{dv}{dr} - \frac{dD}{dr} \\
&= D^2 - \frac{dD}{dr}
\end{aligned}$$

■

Example 3.6 *Let b be a zero coupon bond, with only one payment x occuring at time $t = n$. What is the convexity of bond b?*

Solution 3.6 *From Example 3.4 we have $D = n$.*

$$\begin{aligned}
C &= D^2 - \frac{dD}{dr} \\
&= n^2.
\end{aligned}$$

That worked out nicely!

Example 3.7 *Let b be an perpetuity, with equal payments $x_i = x$ for $i \in \{1, 2, 3...\}$, occuring at time $t_i = i$. In words, there is a constant payment x at the end of each year. The British government once issed perpetuities as their main source of funding. These bonds were called Consols. What is the convexity of b?*

Solution 3.7 *From Example 3.5 we have $D = \dfrac{1}{1 - e^{-r}}$.*

$$\begin{aligned}
C &= D^2 - \frac{dD}{dr} \\
&= \left[\frac{1}{1 - e^{-r}}\right]^2 - \frac{d}{dr}\left[\frac{1}{1 - e^{-r}}\right] \\
&= \left[\frac{1}{1 - e^{-r}}\right]^2 - \frac{-e^{-r}}{(1 - e^{-r})^2} \\
&= \frac{1 + e^{-r}}{(1 - e^{-r})^2}.
\end{aligned}$$

Note that this solution can be approximated by $C \approx \dfrac{2 - r}{r^2}$ when r is small.

Immunization

Figure 3.8 presents the present values of a perpetuity and a zero coupon bond. The maturity of the zero coupon bond is such that the duration of the two bonds are equal when the present values are equal. The perpetuity is more convex than the zero coupon bond. The downside risk is smaller for the perpetuity than for the zero coupon bond. The upside risk is greater for the perpetuity than for the zero coupon bond.

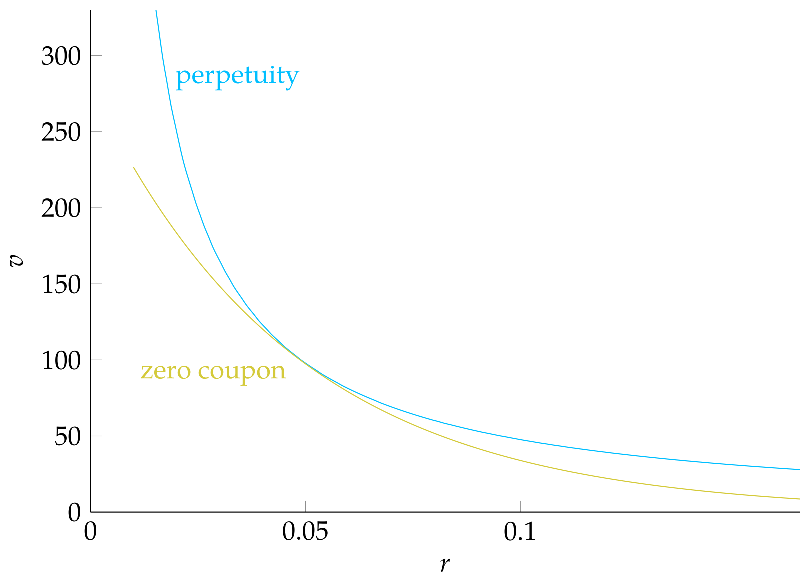


Figure 3.8: Discount rates and present values of perpetuity with annual coupon of 5 and zero coupon instrument with equal present value and duration at discount rate ($r = 0.05$).

Consider an insurance company with long dated liabilities and long dated assets. Ideally, this insurance company would have a portfolio of assets with present value exceeding the present value of the liabilities. Also, the insurance company would wish to match the duration of the portfolio of assets with the portfolio of liabilities. When this is achieved, and the portfolio of assets is more convex than the portfolio of liabilities, the portfolio is said to be immunized against interest rate risk. Increases in the discount rate reduce the present value of liabilities further than they reduce the present value of assets. Decreases in the discount rate increase the present value of liabilities by less than they increase the present value of the assets.

Negative convexity

Consider a *callable bond* with coupon of 5, face value of 100 and a strike price of 100. That is, the borrower can re-purchase the bond at any time for 100. Figure 3.9 presents a plot of present values for this bond. When the present value of the bond rises above the strike price, the borrower will *call* the bond, repurchasing it for 100 (possibly financing this with a new bond issue). This option effectively caps the present value of the bond. Near the strike price, the shape of the present values becomes concave. We refer to this as *negative convexity*. The holder of the bond suffers the downside risk of increased discount rates, but they do not enjoy the upside risk of decreases in discount rates.

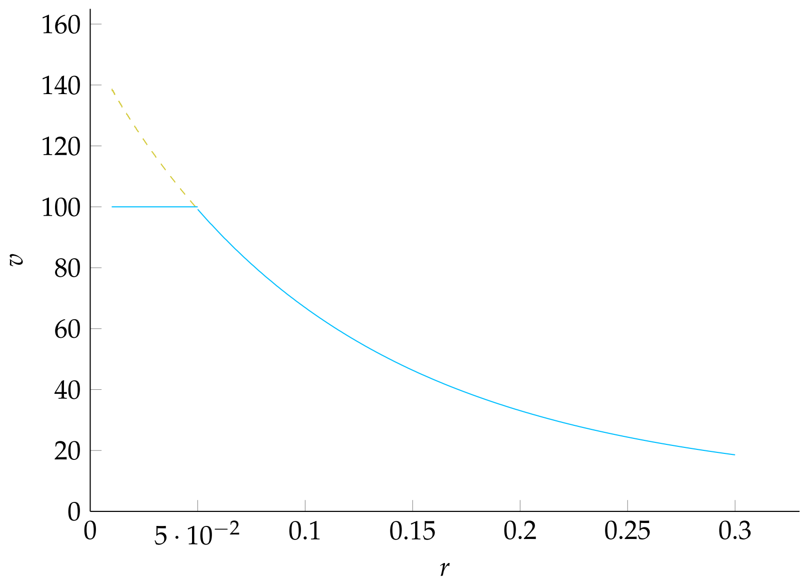


Figure 3.9: Present value of a 10 year callable bond with annual coupon of 5, face value of 100 and strike price of 100. The dashed schedule traces the present values for a non-callable bond.

US mortgages bear this feature. Standard US mortgages are 30 year fixed rate loans with a prepayment option---if interest rates fall, borrowers can seek a new mortgage at a lower rate, prepaying the original loan without penalty. Before the 2007-08 financial crisis, this was considered quite a big problem for the US Mortgage Backed Security market.

In the previous sections, we only considered bonds with fixed, known future payments. The callable bonds considered in this section have risky cash flows that depend on both fluctuations in interest rates and the actions of the borrower. If we have a large portfolio of bonds with uncorrelated credit risk, or if we have a portfolio of safe government bonds, then the assumption of fixed future cash flows is fairly reasonable. When we have a portfolio of risky bonds with correlated risk, or a portfolio of bonds with embedded call options, or a portfolio of bonds with floating interest rates, then the cash flows of our portfolio will be correlated with our discount rates. When we are taking into account the correlation between discount rates and future cash flows, we can generalise the concept of duration.

Effective duration is defined as the semi-elasticity of present value to changes in discount rates, taking into account correlations between discount rates and cash flows. Consider the callable bond from this section. When the discount rate is sufficiently low, the effective duration of the bond becomes zero: further falls in the discount rate encourage the borrower to call the bond, changing the cash flows of the bond and limiting the potential upside risk for the present value of the bond.

Problems for Chapter 3

Exercise 3.1 *Given the following information about bond b, calculate the bond's present value v and duration D.*

Payment 1	$t_1 = 1.5$	$x_1 = 40$
2	$t_2 = 2$	$x_2 = 50$
3	$t_3 = 2.5$	$x_3 = 20$

The (continuously compounded) discount rate is 3%.

Exercise 3.2 *You have the following information about the present values and discount rates for a bond b:*

r	v
0%	105
2%	104
4%	95

a. *Use this information to calculate a numerical estimate of duration for the bond, evaluated at* $r = 2\%$.

b. *Use this information to calculate a numerical estimate of convexity for the bond, evaluated at* $r = 2\%$.

c. *What does your answer to (b) tell you about the terms of the bond, if anything?*

Exercise 3.3 *You are given the following formula to derive the duration of a bond, where x is the annual coupon of the bond, f is the face value of the bond, n is the maturity of the bond and r is the continuously compounding discount rate.*

$$D = \frac{\left(\frac{1}{1-e^{-r}} - \frac{n}{e^{rn}-1}\right) x + \frac{e^r - 1}{e^{rn}-1} n f}{x + \frac{e^r - 1}{e^{rn}-1} f}$$

From this formula, derive formulas for the duration of

a. *a zero coupon bond,*

b. *an annuity,*

c. *a perpetuity.*

Exercise 3.4 *Explain the concept of immunization.*

Exercise 3.5 *Use the concepts of duration and convexity to describe why banks may be reluctant to issue long term fixed interest rate loans when interest rates are low.*

Exercise 3.6 *a. Consider a perpetuity and a zero coupon bond with equal present value and duration. Prove that the perpetuity has greater convexity.*

b. Describe in words why a perpetuity has greater convexity than a zero coupon bond with equal present value and duration.

Exercise 3.7 *What is the effective duration of a stock?*

Selected solutions for Chapter 3

Exercise Solution 3.1 *First, we'll take a look at the bond and determine some general characteristics. This will tell us how to approach the problem and to establish initial guesses about the present value and duration of the bond.*

The bond has three payments, at 18 months, 24 months and 30 months from now. The payments are not constant over time (the payments are 40, 50 and 20). It follows that the bond is not a zero coupon bond, it is not an annuity, and it is not a standard coupon bond. So, we cannot use the standard formulas presented within the lecture.

Fortunately, as there are only three payments, it is pretty straightforward to solve for present value and duration "the long way".

Present value

First lets establish bounds. The total amount of cash flows is

$$x_1 + x_2 + x_3 = 40 + 50 + 20 = 110.$$

So, we would expect the present value of the bond to be positive but less than 110. To solve for the present value of the bond, we sum the present values of each payment:

$$\begin{aligned} v &= v_1 + v_2 + v_3 \\ &= \exp(-rt_1)x_1 + \exp(-rt_2)x_2 + \exp(-rt_3)x_3 \\ &= \exp(-0.03 \times 1.5)40 + \exp(-0.03 \times 2)50 + \exp(-0.03 \times 2.5)20 \\ &= 38.24 + 47.09 + 18.55 \qquad \textit{(Record these values for the duration calculations.)} \\ &= 103.88. \end{aligned}$$

We should check that this answer is reasonable given our initial upper bound of 110. It is.

Duration

It would be helpful to avoid taking derivatives. By Proposition 3.2, we know that the duration of a bond is equal to the present value weighted average of the timing of payments.

Again, first lets establish bounds. The first payment is at time 1.5, and the second payment is at time 2.5. These timings provide lower and upper bounds on the duration of the bond.

The present value weighted average of the timing of payments can be *expressed as follows:*

$$
\begin{aligned}
D &= \sum_{i=1}^{n} \frac{v_i}{v} t_i \\
&= \frac{v_1}{v} t_1 + \frac{v_2}{v} t_2 + \frac{v_3}{v} t_3 \\
&= \frac{38.24}{103.88} 1.5 + \frac{47.09}{103.88} 2 + \frac{18.55}{103.88} 2.5 \\
&= 0.55 + 0.91 + 0.45 \\
&= 1.91.
\end{aligned}
$$

This answer is consistent with our upper and lower bounds of 1.5 and 2.5.

Exercise Solution 3.2

a. By Definition 3.1, Duration is given by

$$D = \frac{-1}{v} \frac{dv}{dr}.$$

In this example, we are not given functional forms, and cannot take derivatives directly. Instead, we need to approximate $\frac{dv}{dr}$ *using finite difference methods. We can use the finite difference formula*

$$\frac{dv}{dr} \approx \frac{v(x + \frac{1}{2}h) - v(x - \frac{1}{2}h)}{h}.$$

We need to set x *and* h *such that we can use the information given in the question. Specifically, we need* $x + \frac{1}{2}h = 4\%$*, and* $x - \frac{1}{2}h = 0\%$*. It follows that* $x = 2\%$*, and* $h = 4\%$*.*

We can re-write the formula as follows

$$\frac{dv}{dr} \approx \frac{v(r = 4\%) - v(r = 0\%)}{4\%}.$$

Note that $v(r = 0\%)$ *represents the present value evaluated at discount rate* $r = 0\%$*.*

$$
\begin{aligned}
\frac{dv}{dr} &\approx \frac{95 - 105}{0.04} \\
&\approx -\frac{10}{0.04}.
\end{aligned}
$$

We are asked to find the duration at $r = 2\%$*. Therefore, the present value is*

$$v = 104.$$

Now, we use these values to solve for duration

$$
\begin{aligned}
D &= \frac{-1}{v} \frac{dv}{dr} \\
&= \frac{-1}{104} \times \left(\frac{-10}{0.04} \right) \\
&= 2.40.
\end{aligned}
$$

In words, we would expect the value of the bond to fall by 2.4% following a 1% increase in discount rates.

b. *By Definition 3.2, Duration is given by*

$$C = \frac{1}{v}\frac{d^2v}{dr^2}.$$

In this example, we are not given functional forms, and cannot take derivatives directly. Instead, we need to approximate $\frac{d^2v}{dr^2}$ *using finite difference methods. We can use the finite difference formula*

$$\frac{d^2v}{dr^2} \approx \frac{v(x+h) - 2v(x) + v(x-h)}{h^2}.$$

Again, we need to set x and h such that we can use the information given in the question. Specifically, we need $x + h = 4\%$*, and* $x - h = 0\%$*. It follows that* $x = 2\%$*, and* $h = 2\%$*.*

We can re-write the formula as follows

$$\frac{dv}{dr} \approx \frac{v(r=4\%) - 2v(r=2\%) + v(r=0\%)}{(2\%)^2}.$$

Note that $v(r = 0\%)$ *represents the present value evaluated at discount rate* $r = 0\%$*.*

$$\begin{aligned}\frac{dv}{dr} &\approx \frac{95 - 2(104) + 105}{0.02^2} \\ &\approx -\frac{8}{0.0004}.\end{aligned}$$

We are asked to find the convexity at $r = 2\%$*. Therefore, the present value is*

$$v = 104.$$

Now, we use these values to solve for convexity

$$\begin{aligned}C &= \frac{1}{v}\frac{d^2v}{dr^2} \\ &= \frac{1}{104} \times \left(\frac{-8}{0.0004}\right) \\ &= -192.\end{aligned}$$

c. *This bond has negative convexity. This may mean that the bond is callable by the issuer.*

Some sums

Property 3.1

$$\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$$

Proof.

$$\begin{aligned}
\sum_{i=0}^{n} x^i &= 1 + x + x^2 + ... + x^n \\
&= \frac{(1-x)(1+x+x^2+...+x^n)}{1-x} \\
&= \frac{1(1-x) + x(1-x) + x^2(1-x) + ... + x^n(1-x)}{1-x} \\
&= \frac{1 - x + x - x^2 + x^2 - x^3 + ... + x^n - x^{n+1}}{1-x} \\
&= \frac{1-x^{n+1}}{1-x}
\end{aligned}$$

■

Property 3.2

$$\sum_{i=1}^{n} x^i = \frac{x(1-x^n)}{1-x}$$

Proof.

$$\begin{aligned}
\sum_{i=1}^{n} x^i &= x + x^2 + x^3 + ... + x^n \\
&= x(1 + x + x^2 + ... + x^{n-1}) \\
&= x \sum_{i=0}^{n-1} x^i \\
&= \frac{x(1-x^n)}{1-x} \qquad \text{by Property 3.1.}
\end{aligned}$$

■

Property 3.3 *If* $x \in (-1, 1)$,

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

Proof. The easy way is to just take the limit of Property 3.1 as n approaches ∞. But I think the following way is particularly clever:

$$\begin{aligned}\sum_{i=0}^{\infty} x^i &= 1 + x + x^2 + x^3 + \ldots \\ &= 1 + x(1 + x + x^2 + x^3 + \ldots) \\ &= 1 + x\sum_{i=1}^{\infty} x^i\end{aligned}$$

Subtract $x\sum_{i=1}^{\infty} x^i$ from both sides,

$$(1-x)\sum_{i=1}^{\infty} x^i = 1.$$

Now divide both sides by $(1-x)$,

$$\sum_{i=1}^{\infty} x^i = \frac{1}{1-x}.$$

■

Property 3.4 *If $x \in (-1, 1)$,*

$$\sum_{i=1}^{\infty} x^i = \frac{x}{1-x}$$

Proof. We just want to find a way to apply Property 3.3. One approach is to factor out x,

$$\begin{aligned}\sum_{i=1}^{\infty} x^i &= x + x^2 + x^3 + \ldots \\ &= x(1 + x + x^2 + x^3 + \ldots) \\ &= x\sum_{i=0}^{\infty} x^i \\ &= \frac{x}{1-x}.\end{aligned}$$

Alternatively, we can just add and subtract 1,

$$\begin{aligned}\sum_{i=1}^{\infty} x^i &= x + x^2 + \ldots \\ &= -1 + 1 + x + x^2 + \ldots \\ &= -1 + \sum_{i=0}^{\infty} x^i \\ &= \frac{1}{1-x} - 1 \\ &= \frac{x}{1-x}.\end{aligned}$$

■

Property 3.5 *If* $x \in (-1, 1)$,

$$\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}.$$

Proof. As usual, we start by writing out the sum:

$$\sum_{i=1}^{\infty} ix^i = x + 2x^2 + 3x^3 + \ldots$$

The trick with this one is to recognise that it *almost* looks like a derivative. If we factor out x then we get

$$\sum_{i=1}^{\infty} ix^i = x(1 + 2x + 3x^2 + \ldots)$$

The term in brackets is indeed a derivative, so lets see if that helps us solve the problem:

$$\begin{aligned}
\sum_{i=1}^{\infty} ix^i &= x\frac{d}{dx}\left[x + x^2 + x^3 + \ldots\right] \\
&= x\frac{d}{dx}\left[\sum_{i=1}^{\infty} x^i\right] \\
&= x\frac{d}{dx}\left[\frac{x}{1-x}\right] \\
&= x\left[\frac{1}{(1-x)^2}\right] \\
&= \frac{x}{(1-x)^2}.
\end{aligned}$$

By Property 3.4

By the quotient rule, if

$$f(x) = g(x)/h(x), \quad \text{then}$$

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}.$$

■

4
Forwards

It is difficult to get the news from poems yet men die miserably every day for lack of what is found there.
William Carlos Williams

Introduction and overview

Lets start with an excerpt from a Stephen Dubner interview of Ray Dalio on the Freakonomics podcast.[1]

DUBNER: Can you tell me briefly how you helped McDonald's launch the McNugget?
DALIO: Well, okay — I graduated from business school in 1973, and I traded commodities, and I love to trade commodities, and I love the mechanics of it. There was something about — you can grow a chicken and it's so many pounds of this and that that makes the chicken come around and blah blah blah. And I had two clients at the time: McDonald's and a chicken producer. And McDonald's wanted to come out with the McNuggets. But there was a lot of volatility in the chicken market at that time and they were worried that if they set a menu price and the price of chicken then went through the roof that they would get squeezed or they'd have to raise the prices and it would be unstable.
And were they worried that their introduction of the product was going to spike demand or spike price because of their action because they're that big? Or orthogonal to that.
They were just worried that the cost of the chicken would go up. But there was not a way for them to hedge that, because there was not an adequate chicken market. But the producer of the chickens — since a chicken is mostly a little chick and then it has a lot of grain that's added, and you could use the futures market — what I did is I showed him how we can hedge his cost and that he could provide a fixed price to McDonald's for chicken McNuggets.
He could hedge his costs by buying corn or buying or selling corn and soybean futures then, is that the idea?
Yeah, corn and soybean meal futures because that was where his volatility was. He could lock it through. And so by doing that we engineered that. I don't know how interesting it is, but it was an engineering exercise.

In general, a forward contract will help agents solve the following problem: Bernie wishes to lock in an asset price at time t for delivery of asset X at time T. Sam agrees to deliver the asset at specified time T. The price agreed at time t for this contract is denoted $F(X, t, T)$. Lets look at some examples:

[1] The full interview available here `http://freakonomics.com/podcast/ray-dalio/` [accessed 9 April 2018].

Bonds

Example 4.1 *Let $B_{t,t+2}$ be a two year risk free bond issued at time t and maturing at time $t+2$. Bond $B_{t,t+2}$ has a face value of $FV(B_{t,t+2}) = 100$, and pays coupon annually with coupon rate $r_{t,t+2} = 5\%$. The market price of bond $B_{t,t+2}$ at time t is $P_t(B_{t,t+2}) = 100$. Let $B_{t,t+1}$ be a one year risk free bond issued at time t and maturing at time $t+1$. Bond $B_{t,t+1}$ has a face value of $FV(B_{t,t+1}) = 100$, and pays coupon annually with coupon rate $r_{t,t+1} = 7\%$. The market price of bond $B_{t,t+1}$ at time t is $P_t(B_{t,t+1}) = 100$.*

What is the arbitrage-free ex-coupon price $F(B_{t,t+2}, t, t+1)$?

Solution 4.1 *We'll start by considering the cash flows of the two bonds.*

The cash flows for bonds $B_{t,t+2}$ and $B_{t,t+1}$ are

Figure 4.1: Bond $B_{t,t+2}$

Period		
$t+0$	$t+1$	$t+2$
-100	5	105
$P_t(B_{t,t+2})$	$r_{t,t+2} \times FV(B_{t,t+2})$	$FV(B_{t,t+2}) + (r_{t,t+2} \times FV(B_{t,t+2}))$
Initial price	Coupon	Coupon plus principal

Figure 4.2: Bond $B_{t,t+1}$

Period	
$t+0$	$t+1$
-100	107
$P_t(B_{t,t+1})$	$FV(B_{t,t+1}) + (r_{t,t+1} \times FV(B_{t,t+1}))$
Initial price	Coupon plus principal

What we want to do is construct a strategy, using bond $B_{t,t+1}$ and the forward, $F(B_{t,t+2}, t, t+1)$, that replicates the payoffs of the bond $B_{t,t+2}$. In this example, the strategy is quite simple. At time zero, we buy the one year bond $B_{t,t+1}$ and the forward $F(B_{t,t+2}, t, t+1)$. At time 1, we receive coupon and principal on the one year bond, we pay the forward price and we receive the two year bond (ex coupon). At time 2, we receive the final coupon bond and principal of the two year bond.

Figure 4.3: Bond $B_{t,t+1}$ and forward contract

Period		
$t+0$	$t+1$	$t+2$
-100	$107 - F(B_{t,t+2}, t, t+1)$	105
$P_t(B_{t,t+2})$	$(1 + r_{t,t+1}) \times FV(B_{t,t+1}) - F(B_{t,t+2}, t, t+1)$	$(1 + r_{t,t+2}) \times FV(B_{t,t+2})$
Initial price	Coupon	Coupon plus principal

This strategy replicates the cash flows of the two year bond when the forward price $F(B_{t,t+2}, t, t+1)$ solves

$$107 - F(B_{t,t+2}, t, t+1) = 5.$$

That is, when $F(B_{t,t+2}, t, t+1) = 102$, which is the arbitrage-free ex-coupon forward price.

Yield curves

Example 4.2 *Consider example 4.1. What is the arbitrage free forward interest rate of a bond issued at time $t+1$, $R(t+1, t+2)$?*

Solution 4.2 *The strategy described in Figure 4.3 involves the forward purchase at time $t+1$ of a bond maturing at time $t+2$. We can derive the arbitrage free forward interest rate from the return to this bond.*

The forward price of the bond is $F(B_{t,t+2}, t, t+1) = 102$. There is only one payoff for this bond, at time $t+2$, of 105. The forward interest rate $R(t+1, t+2)$ is the effective annual rate of this hypothetical bond:

$$R(t+1, t+2) = 105/102 - 1 = 2.94\%$$

Foreign exchange

Example 4.3 *Let $r(\textsc{gbp}, t, t+1) = 7\%$ be the interest rate on a one year riskless bond issued at time t in* GBP. *Let $r(\textsc{usd}, t, t+1) = 5\%$ be the interest rate on a one year riskless bond issued at time t in* USD. *The current exchange rate is $P(\textsc{gbp}/\textsc{usd}, t) = 1.23$. That is, 1 British pound equals 1.23 US dollars. What is the arbitrage-free forward exchange rate $F(\textsc{gbp}/\textsc{usd}, t, t+1)$?*

Solution 4.3 *Similar to the previous example, we want to construct two portfolios with replicating payoffs, where one of the portfolios uses the forward. The arbitrage-free forward price is the price that ensures that payoffs are the same across the strategies.*

We'll consider the following two strategies:

Strategy 1: Start with GBP *1. Exchange this into* USD*. Purchase the* USD *bond.*

Strategy 2: Start with GBP *1. Purchase the* GBP *bond* and $(1 + r(\textsc{gbp}, t, t+1))$ *units of the forward contract.*

Figure 4.4 describes these two strategies. The rows differentiate between currencies, the columns differentiate time periods. Strategy 1 is described by the blue arrows; first the currency is exchanged then the bond is purchased. Strategy 2 is described by the red arrows; first the bond is purchased and then the currency is exchanged.

Strategy 1 *Starting with* GBP *1, we exchange this into* USD*:*

$$[\textsc{gbp}1] \times P(\textsc{gbp}/\textsc{usd}, t) = \textsc{usd}\ 1.23.$$

With the USD 1.23*, we buy 1.23 units of the* USD *bond at time t. At time $t+1$, we have*

$$\textsc{usd}\ 1.23 \times (1 + r(\textsc{usd}, t, t+1)) = 1.23 \times 1.05 = \textsc{usd}\ 1.29.$$

Figure 4.4: Foreign exchange forwards

$$\begin{tikzcd}
\textsc{gbp}, t \arrow[r, "1+r(\textsc{gbp},t,t+1)"] \arrow[d, "P(\textsc{gbp}/\textsc{usd},t)"'] & \textsc{gbp}, t+1 \arrow[d, "F(\textsc{gbp}/\textsc{usd},t,t+1)"] \\
\textsc{usd}, t \arrow[r, "1+r(\textsc{usd},t,t+1)"'] & \textsc{usd}, t+1
\end{tikzcd}$$

Strategy 2 *Starting with* GBP 1*, we purchase the* GBP *bond and* $1 + r(\textsc{gbp}, t, t + 1)$ *units of the forward contract. When the bond matures at time* $t + 1$*, we have*

$$1 + r(\textsc{gbp}, t, t + 1) = \textsc{gbp}\ 1.07.$$

This converts into $F(\textsc{gbp}/\textsc{usd}, t, t + 1) \times [\textsc{gbp}\ 1.07]$ *units of* USD.

These two strategies have the same payoffs if and only if the final amount of USD *held in each strategy is the same. In other words,*

$$F(\textsc{gbp}/\textsc{usd}, t, t + 1) \times [\textsc{gbp}\ 1.07] = \textsc{usd}\ 1.29$$

$$\begin{aligned} F(\textsc{gbp}/\textsc{usd}, t, t + 1) &= \frac{1.29}{1.07} \\ &= 1.21. \end{aligned}$$

Lets add the numbers from our example:

Figure 4.5: Foreign exchange forwards example

$$\begin{tikzcd}
\textsc{gbp}\ 1, t \arrow[r, "1+r(\textsc{gbp},t,t+1)=1.07"] \arrow[d, "P(\textsc{gbp}/\textsc{usd},t)=1.23"'] & \textsc{gbp}\ 1.07, t+1 \arrow[d, "F(\textsc{gbp}/\textsc{usd},t,t+1)=1.21"] \\
\textsc{usd}\ 1.23, t \arrow[r, "1+r(\textsc{usd},t,t+1)=1.05"'] & \textsc{usd}\ 1.29, t+1
\end{tikzcd}$$

Problems for Chapter 4

Exercise 4.1 *The current (time t) exchange rate to convert British pounds into Australian dollars is $P(\text{GBP}/\text{AUD}, t) = 1.72$. That is, 1 British pound will buy 1.72 Australian dollars. The current GBP interest rate on 10-year government bonds is 1.08% per annum. The current AUD interest rate on 10-year government bonds is 2.57% per annum.*

Show that the arbitrage-free ten year forward exchange rate $F(\text{GBP}/\text{AUD}, t, t+10) = 1.99$.

Exercise 4.2 *The current (time t) exchange rate to convert British pounds into US dollars is $P(\text{GBP}/\text{USD}, t) = 1.29$. That is, 1 British pound will buy 1.29 US dollars. The current GBP interest rate on 10-year government bonds is 1.08% per annum. The current USD interest rate on 10-year government bonds is 2.29% per annum.*

Show that the arbitrage-free ten year forward exchange rate $F(\text{GBP}/\text{EUR}, t, t+10) = 1.45$.

Exercise 4.3 *The current (time t) exchange rate to convert British pounds into euros is $P(\text{GBP}/\text{EUR}, t) = 1.18$. That is, 1 British pound will buy 1.18 euros. The current GBP interest rate on 10-year government bonds is 1.08% per annum. The current EUR interest rate on 10-year (German) government bonds is 0.31% per annum.*

Show that the arbitrage-free ten year forward exchange rate $F(\text{GBP}/\text{EUR}, t, t+10) = 1.09$.

Exercise 4.4 *Let $B_{t,t+2}$ be a two year risk free bond issued at time t and maturing at time $t+2$. Bond $B_{t,t+2}$ has a face value of $FV(B_{t,t+2}) = 100$, and pays coupon annually with coupon rate $r_{t,t+2} = 2.30\%$. The market price of bond $B_{t,t+2}$ at time t is $P_t(B_{t,t+2}) = 100$. Let $B_{t,t+1}$ be a one year risk free bond issued at time t and maturing at time $t+1$. Bond $B_{t,t+1}$ has a face value of $FV(B_{t,t+1}) = 100$, and pays coupon annually with coupon rate $r_{t,t+1} = 1.40\%$. The market price of bond $B_{t,t+1}$ at time t is $P_t(B_{t,t+1}) = 100$.*

Show that the arbitrage-free ex-coupon price $F(B_{t,t+2}, t, t+1)$ is 99.10.

5
Efficiency

A dungeon horrible, on all sides round,
as one great furnace flamed; yet from those flames
no light; but rather darkness visible
served only to discover sights of woe,
regions of sorrow, doleful shades, where peace
and rest can never dwell, hope never comes
that comes to all, but torture without end
still urges, and a fiery deluge, fed
with ever-burning sulphur unconsumed.
Such place Eternal Justice had prepared
for those rebellious; here their prison ordained
in utter darkness, and their portion set,
as far removed from God and light of Heaven
as from the centre thrice to the utmost pole.
Oh how unlike the place from whence they fell!
John Milton, *Paradise Lost*, Book I, 61-75.

Introduction

> One thing we are not going to have, now or ever, is a set of models that forecasts sudden falls in the value of financial assets, like the declines that followed the failure of Lehman Brothers in September. ... The main lesson we should take away from the [Efficient Market Hypothesis] for policymaking purposes is the futility of trying to deal with crises and recessions by finding central bankers and regulators who can identify and puncture bubbles. If these people exist, we will not be able to afford them.
> Robert Lucas, 2009.

Asset market efficiency

You will often hear comentators and academics discuss and critique the *Efficient Market Hypothesis* (EMH). But the EMH in itself is quite narrow in its application, referring to the extent to which new information is incorporated in asset prices.

What we as economists really care about, or should care about, is *allocative efficiency*, which in finance means that savings are being mobilised into productive investments, and that savers can share the rewards from those investments.

The two forms of efficiency are related, and departures from the EMH will result in departures from allocative efficiency. But, its important to keep in mind that the two concepts are distinct, and what matters for economic growth and welfare is allocative efficiency.

In the rest of this section, we describe these two concepts of asset market efficiency in more detail, along with other useful measures of asset market efficiency.

Allocative efficiency

Allocative efficiency is the broadest and most general form of efficiency that we consider here. The primary roles of financial markets are (1) to allocate risk and (2) to allocate consumption over time.

Let $\mathrm{MRS}^i_{a,b}$ refer to the consumption marginal rate of substitution of agent i across periods and states a, b. Let $\mathrm{MRT}_{a,b}$ refer to the economy's production marginal rate of transformation across periods and states a, b. Under perfect markets, efficient allocations of risk require that the following holds

$$\mathrm{MRS}^i_{z,z'|t} = \mathrm{MRS}^j_{z,z'|t} = \mathrm{MRT}_{z,z'|t}, \tag{5.1}$$

where z and z' are different states of the world at a given date t.

In words, for all agents, the rate at which they would forego consumption in one state of the world for consumption in another state of the world must equal the rate at which the economy can transform

Remember, the *marginal rate of substitution* from apples to oranges is the amount of apples that you would give up for one orange. The *marginal rate of transformation* from apples to oranges is the amount of apples an orchadist would forego per orange grown if they were to replace apple trees with orange trees.

savings in one state of the world into consumption in another state of the world.

Also, under perfect markets, efficient allocations of consumption over time require that the following holds

$$\mathbb{E}[\mathrm{MRS}^{i}_{t,t+1}] = \mathbb{E}[\mathrm{MRS}^{j}_{t,t+1}] = \mathbb{E}[\mathrm{MRT}_{t,t+1}].$$

In words, for all agents, the rate at which they would forego consumption today for consumption tomorrow must equal the rate at which the economy can transform savings today into future consumption.

We can combine these two conditions as follows:

$$\mathrm{MRS}^{i}_{t-1,t} = \mathrm{MRS}^{j}_{t-1,t} = \mathrm{MRT}_{t-1,t}.$$

Allocative efficiency requires that all projects with equal risks and payoffs should trade at the same price. On other words, allocative efficiency requires that there are no unexploited opportunities for arbitrage. An explanation of this result is left as an exercise for students.

Example 5.1 *Using equation 5.1, describe the efficient allocation of the risk that your bike is stolen.*

Solution 5.1 *Let z' refer to the state of the world in which your bike is stolen. In state z, your bike is not stolen. All else equal, and in lieu of insurance, we would expect that your consumption marginal utility should be high when your bike is stolen relative to when your bike is not stolen. That is, without insurance, $MRS_{z,z'|t} > 1$. In words, you have a greater desire for additional income in states where your bike is stolen than in states when your bike is not stolen---you could use this income to buy a new bike or to cover public transport costs.*

Consider another individual Jamie, (j). Jamie's bike is not stolen in either state, Jamie is therefore indifferent between additional income in states z, z'. Mathematically, $MRS^{j}_{z,z'|t} = 1$. Additionally, the production capability of the economy is unaffected by the stealing of your bike. So the production marginal rate of transformation is $MRT_{z,z'|t} = 1$.

According to Equation 5.1, the efficient allocation of risk requires that $MRS_{z,z'|t} = MRS^{j}_{z,z'|t} = MRT_{z,z'|t}$. This can be achieved with an insurance contract offered by other agents in the economy, rewarding you with a lump sum payout in the state that your bike is stolen. This would reduce your consumption marginal rate of substitution $MRS_{z,z'|t}$ until $MRS_{z,z'|t} = MRS^{j}_{z,z'|t} = MRT_{z,z'|t}$. In a large economy, the cost of this insurance can be fully diversified, leaving $MRS_{z,z'|t} = MRS^{j}_{z,z'|t} = MRT_{z,z'|t} = 1$

Example 5.2 *Using equation 5.1, describe the efficient allocation of business cycle risk.*

Solution 5.2 *A recession is a state of the world in which the cost of production of goods is relatively high, $MRT_{z,z'|t} > 1$, where z' is a recession and z is a boom. It follows that the efficient allocation of risk requires that*

$$MRS^i_{z,z'|t} = MRS^j_{z,z'|t} = MRT_{z,z'|t} > 1.$$

In words, under the efficient allocations, all agents would wish to exchange some boom consumption for recession consumption. This allocation can be thought of as sharing the pain of recessions across agents---we're all in this together!

Operational efficiency

Operational efficiency refers to firms' ability to produce goods and services at the lowest possible cost. In financial markets, operational efficiency concerns may relate to market structure, monopoly, and principal agent problems between investors and financial professionals. Regulations can promote operational efficiency by encouraging competition, limiting monopoly pricing power and reducing principal agent costs. On the other hand, where regulations impose costs without compensating benefit, these regulations would decrease operational efficiency.

Portfolio efficiency

An efficient portfolio is one with the least possible variance, holding expected return constant. We return to portfolio efficiency in Chapter 9

Informational efficiency

? writes ``A market in which prices always `fully reflect' available information is called `efficient'.'' This definition is quite far from giving us an empirical test of efficiency. What does it mean for prices to `fully reflect' information? What does it mean for information to be `available'? **?** offers a more formal definition:

The *Efficient Market Hypothesis* states that financial markets are informationally efficient.

Definition 5.1 *A capital market is said to be* efficient *with respect to some information set* Ω*, if security prices would be unaffected by revealing the information set* Ω *to all participants.*

Moreover, efficiency with respect to an information set implies that it is impossible to make *economic profits* by trading on the basis of that information set. The term *economic profit*, rather than *accounting profit*, is important here, and poses a challenge to empirical tests of efficiency.

The term *information set* should be interpreted literally---it is just a set of pieces of information. It is a bit abstract, but it is a helpful way to formalise *beliefs*.

Starting from Definition 5.1, we can further characterize specific forms of market efficiency conditional upon restrictions over the information set Ω:

Definition 5.2

Weak-Form Efficiency*: The market is efficient with respect to an information set Ω_P consisting of the histories of asset prices and returns.*[1]

Semi-Strong-Form Efficiency*: The market is efficient with respect to an information set Ω_{pub} consisting of all* publicly available *information.*

Strong-Form Efficiency*:The market is efficient with respect to an information set $\bar{\Omega}$ consisting of all information known to any market participant.*

We can see from Definition 5.2 the challenges in attempting to define market efficiency. What does it mean for information to be publicly available?

[1] More formally, $\Omega_{\mathbf{p}} = \{\mathbf{p}_t, \mathbf{p}_{t-1}, ...\}$, where $\mathbf{p}_t$ is the vector of asset prices at a given point in time t, that is $\mathbf{p}_t = \{p_t^1, p_t^2, ...\}$.

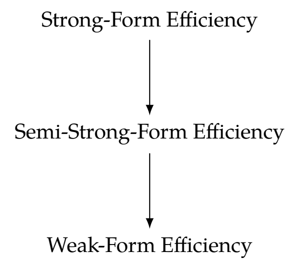


Figure 5.1: The heirachy of market efficiency

Are strong-form efficient markets impossible? The **?** *Paradox*

Consider an asset market that is Strong-Form Efficient with respect to an information set Ω known to all participants at time 0. Let there be a cost for information gathering, which could correspond to time and/or resources.

Trader Terry spends time and resources unearthing a new signal of information, $\omega_T \notin \Omega$. Terry's information set is now $\Omega_T = \{\Omega \cup \omega_T\}$. Let's call this time 1.

It must be the case that at time 0, Terry had the expectation that they would be able to use this additional information ω_T to make an accounting profit. Otherwise, Terry would not have spent the resources and time at time 0 to unearth ω_T. In other words, in terms of Definition 5.1, the market at time 1 is not efficient with respect to Ω_T. As Ω_T is known to market participant Terry, Strong-Form Efficiency cannot hold at time 1.

At time 2, Terry, using the information set Ω_T, buys/sells securities that are under-/overpriced with respect to information set Ω_T. The new equilibrium prices reflect the new information set Ω_T. The new information ω_T is not directly revealed to other market participants, but under the new equilibrium prices, the revelation of ω_T to all market participants would not necessarily have any effect on market prices.

The Grossman-Stiglitz Paradox does not directly preclude Weak-Form or Semi-Strong-Form Efficiency (where *publicly available* is interpreted as available at zero cost). The Paradox does further highlight the subtleties in our definitions of these efficiency benchmarks.

Lesson If information gathering is costly and markets are Strong-Form Efficient, then there is no incentive for traders to gather information. This begs the question of how information was revealed and prices determined in the first place.

The joint hypothesis problem

? says it best:

> The Theory [of asset market informational efficiency] only has empirical content ... within the context of a ... specific model of market equilibrium, that is, a model that specifies the nature of market equilibrium when prices `fully reflect' available information.

We cannot test informational efficiency directly. Informational efficiency on its own does not provide testable predictions.

What we can do, is attempt to derive testable predictions of informational efficiency contingent on a specific model of asset prices, $\mathcal{M} : \Omega \to P$. If testable predictions can be derived for model $\mathcal{M}_i$ then we may be able to test the hypothesis

$$(\mathcal{M} = \mathcal{M}_i) \wedge (\Omega = \Omega_j)$$

If $\mathcal{M}^T$ is the *true model*, then the joint hypothesis test of $(\mathcal{M} = \mathcal{M}^T) \wedge (\Omega = \Omega_j)$ is equivalent to the hypothesis test of $(\Omega = \Omega_j)$, which is what we require to test informational efficiency independently. But, in lieu of the true model, we are stuck with joint hypothesis tests.

Some asset pricing models

In Definition 5.2, we developed three alternative formulations of the information set Ω to be used to test for market efficiency. In Section 5 we argued that to test efficiency with respect to any information set Ω we need to combine this information set with an asset pricing model.

The role of the asset pricing model is to generate econometrically testable predictions from combinations of assumptions including but not limited to the information set available to agents. In this section, we briefly describe a handful of potential models that could be used for this purpose.

Models of efficient markets

The following models were derived from the theory of efficient markets; their predictions vary dependent on the theoretical assumptions underlying the three models. We will study each of these models in later chapters.

Model 5.1 *Martingale Hypothesis.*

$$\mathbb{E}[r_j] = \mu_j \tag{5.2}$$

where μ_j is a constant.

Model 5.2 *Fundamental Valuation Relationship.*

$$\mathbb{E}[r_j] = r_0 - \frac{cov(u'(c), r_j)}{\mathbb{E}[u'(c)]} \tag{5.3}$$

where r_0 is the risk free interest rate, c is consumption, $u'(c)$ is the marginal utility of consumption.

Model 5.3 *Capital Asset Pricing Model (CAPM).*

$$\mathbb{E}[r_j] = r_0 + \beta_j(r_m - r_0), \tag{5.4}$$

where r_m is the return to the market portfolio*, and $\beta_j = \dfrac{cov(r_j, r_m)}{var(r_m)}$*

Atheoretical models

The following models were developed as adaptations of the Capital Asset Pricing Model to better fit historical returns data. These models were not initially derived from the theory of efficient markets, however this does not necessarily mean that they are models of inefficient markets; future researchers may develop models of efficient markets that yield the same predictions.

Model 5.4 *Market Model.*

$$\mathbb{E}[r_j] = r_0 + \alpha_j + \beta_j(r_m - r_0), \tag{5.5}$$

where α_j is a constant, $\beta_j = \dfrac{cov(r_j, r_m)}{var(r_m)}$.

Model 5.5 *Factor Models.*

$$\mathbb{E}[r_j] = r_0 + \sum_{i=1}^{n} \beta_{ji} f_i, \tag{5.6}$$

where the individual random variables f_i, the factors*, can include additional financial information relating to firm j (**?**) as well as macroeconomic indicators including industrial production (**?**).*

Both of the market model and the factor models above are extensions of the CAPM. These extensions were designed as econometric tools to help identify the predictors of stock price returns and guide future theoretical research.

Testing informational efficiency

Autocorrelation studies

The Martingale Hypothesis (Model 5.1) predicts that the best predictor of the future returns of a stock is a constant (this constant represents time preference and risk premia). Importantly, past returns are not predicted not contain any information about expected future returns.

It follows that findings of autocorrelation in historical stock price returns can be considered evidence against the joint hypothesis that (1) markets are weak-form informationally efficient and (2) the martingale hypothesis is the true model of efficient market returns. This autocorrelation could take the form of *mean-reversion*, where poor performing assets tend to outperform the market in subsquent periods, or *momentum*, where poor performing assets tend to underperform the market in subsequent periods.

Early studies of market efficiency, finding both mean-reversion and momentum in the data, typically interpreted these results as being evidence against financial market efficiency. This conclusion is now seen as being too strong. The Fundamental Valuation Relationship (Model 5.2) and the CAPM (Model 5.3) both permit autocorrelation of returns in informationally efficient asset markets under the condition that the covariance between asset returns and consumption or asset returns and market returns respectively exhibit autocorrelation. This is an example of the importance of the joint hypothesis problem. Our choice of asset pricing model can determine whether or not our econometric tests accept or reject market efficiency.

See **?**, Ch.2 Sec.4 and for more information about autocorrelation tests of market informational efficiency. Also, see **?** for an important critique of time series tests of asset market efficiency, and see **?** for an excellent review of the literature on asset pricing anomalies.

Event studies

? provides the first known example of an event study in finance. [2] Dolley sought to discover whether the price of a stock was likely to fall or rise following stock-splits. Stock-splits are just redenominations of stocks, normally undertaken to keep the stock price within a range that facilitates trading. For example, on 9 June 2014, Apple conducted a 7-for-1 stock split. Apple's shareholders woke up to see each of their Apple shares converted into 7 new shares, and the share price fell from \$645.54 before the split to \$92.70 when trading opened after the split.[3]

[2] James Clay Dolley. Characteristics and procedure of common stock split-ups. *Harvard Business Review*, pages 316--326, Apr 1933

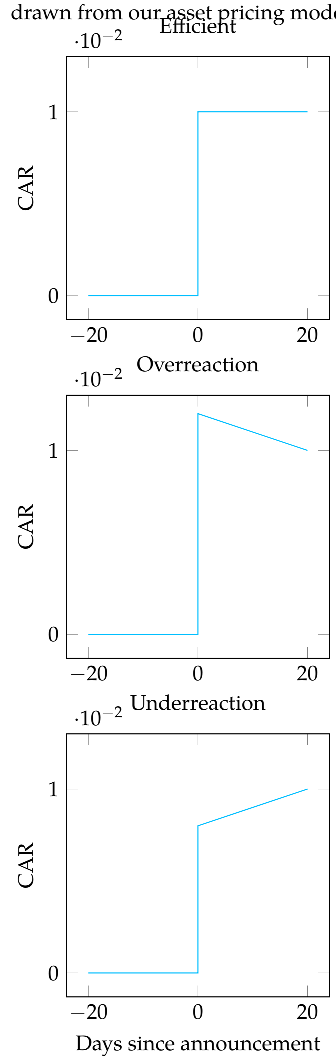
[3] The new price was not exactly 1/7th of the initial price, but there was a passage of time between the close of the day before the stock split and the opening of the day following the stock split.

What Dolley was interested in is whether or not it was profitable on average to purchase (or short sell) stocks following stock-splits. In a semi-strong-form efficient market, we would expect stock prices to adjust instantly to the stock split, and behave otherwise unpredictably both before and after the split.

More generally, event studies compare the returns of securities before and after specified events. An event for this purpose could be an earnings report, a stock split, a merger announcement, or the removal of a CEO. Typically, these events are thought to be included in the publically available information set, and event studies are often used to test semi-strong form market efficiency (See Definition 5.2).

We measure whether or not events give rise to profit opportunities through the concept of *abnormal returns*, the excess return to holding a stock over and above what would be predicted by an asset pricing model.

We use Cumulative Abnormal Returns, rather than just stock prices, to take account of stock-splits, dividends, and changes in expectations of stock prices drawn from our asset pricing model.



Abnormal returns are defined as follows,

$$AR_t = R_t - \mathbb{E}[R_t],$$

where $\mathbb{E}[R_t]$ is generated by an asset pricing model.

The sum of abnormal returns over a period of time is called the *cumulative abnormal return*,

$$CAR_{T,T+n} = \sum_{t=T+1}^{T+n} AR_t.$$

If the event does change the underlying value of the stock, then in an efficient market we would expect cumulative abnormal returns to jump on impact. If the event is good news, we expect that the CAR will jump when the news of the event is released. If the event is bad news, we expect that the CAR will fall when the news of the event is released.

What we don't expect is predictable trends in the CAR following or before the announcement. If the CAR continues to rise after positive events, this would be evidence that the market, on average, underreacts to news of these events; Investors could profit by buying stocks following announcements of similar events. If the CAR falls after positive events, this would be evidence that the market, on average, overreacts to news of these events; Investors could profit by short-selling stocks following announcements of similar events.

Figure 5.2 presents and example of an event study, taken from **?**, Ch.4. They study the effects of quarterly income announcements for 30 companies over 5 years (600 observations total). These announcements are categorised into ``good news'', ``no news'' and ``bad news''. Their results suggest that these announcements do carry information content, and they do elicit price movements in response. For the ``good news'' firms, it does appear that there is some anticipation of the good news implicit in prices before the date of the announcement.

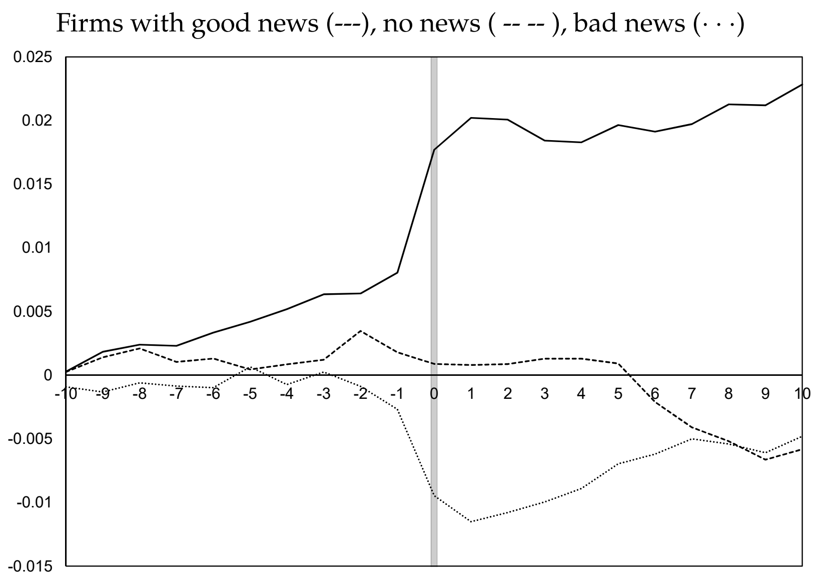


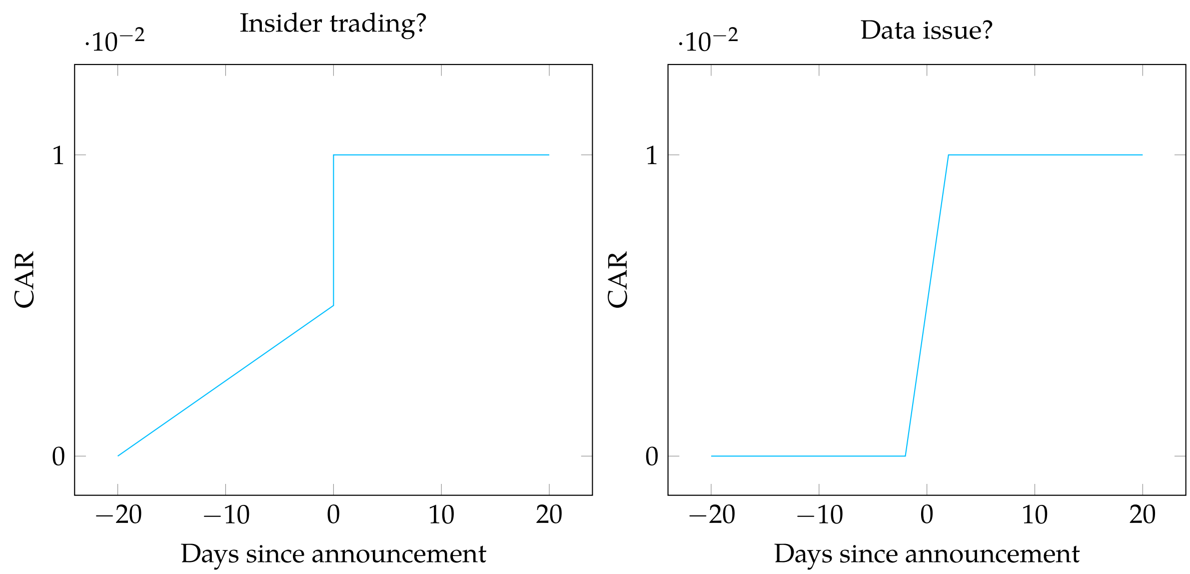
Figure 5.2: Cumulative Abnormal Returns against Days Since Announcement

Aside from over- and underreaction, there are a few other things we should look out for when conducting event studies. If prices start to move before the event, this could be evidence of insider trading---it is possible that some agents are trading on the information before it is publically announced. If the CAR does not jump at time zero but rather starts and finishes moving within a day or two of time zero, this may be evidence of asset market inefficiency, but it also might be the case that the timestamps in your dataset are incorrect and don't match up perfectly with trading days.

Silly things that happen

Sometimes, things happen that fly in the face of the efficient market hypothesis. Consider this example

> Google said on Monday it had agreed to buy Nest Labs, which makes Internet-connected devices like thermostats and smoke alarms, for $3.2 billion in cash. Nest Labs is a private company based in Palo Alto, Calif.
>
> After the deal was announced, investors rushed to buy shares of Nestor, apparently confused by the company's ticker symbol, NEST. The stock continued trading on Wednesday, settling at about 4 cents a share by midday. ...
>
> Nestor Inc. is a defunct shell of a company that once sold automated traffic enforcement equipment to state and local governments. Based in Providence, R.I., the company went into receivership in 2009, and all of its assets have been sold.



Its shares, which are not listed on any exchange, [had] been dormant for years, worth less than a penny each.

New York Times, DealBook[4]

[4] William Alden. A case of mistaken identity sends a worthless stock soaring. URL `https://dealbook.nytimes.com/2014/01/15/a-case-of-mistaken-identity-sends-a-worthl` Accessed 2 August 2018

This example is a clear departure from semi-strong form efficiency. The acquisition information was publicly available, and was misinterpreted by investors who acted on the news but purchased the shares of the wrong company.

Problems for Chapter 5

Exercise 5.1 *Explain why allocative efficiency requires that all projects with equal risks and payoffs should trade at the same price.*

Exercise 5.2 *What do examples 5.1 and 5.2 tell us about what types of risk should offer high returns and what types of risks should not offer high returns in efficient markets?*

Exercise 5.3 *Using an appropriate diagram, plot the efficient allocation of business cycle risk, consistent with Equation 5.1. Your diagram should plot the economy's production possibility frontier across booms and recessions, as well as two different agents' indifference curves across boom and recession consumption.*

6
The Martingale Hypothesis

No man can always have adequate reasons for buying or selling stocks daily — or sufficient knowledge to make his play an intelligent play
Jesse Livermore

Markov processes and martingales

The martingale hypothesis is the first asset pricing model that we will consider.

Definition 6.1 *A discrete time stochastic process X is said to be* Markov process *if*

$$\mathbb{P}(X_t = x_t|\Omega_{t-1}) = \mathbb{P}(X_t = x_t|X_{t-1}) \qquad \textit{where } X^{t-1} \in \Omega_{t-1}$$

(Note that $X^{t-1} = (X_{t-1} = x_{t-1}, ..., X_0 = x_0)$).

Property 6.1 *If X is a Markov process, then*

$$\mathbb{P}(X_t = x_t|X_{t-1} = x_{t-1}, ..., X_0 = x_0) = \mathbb{P}(X_t = x_t|X_{t-1} = x_{t-1})$$

Definition 6.2 *A discrete time stochastic process X is said to be a* martingale *if and only if*

1. $\mathbb{E}(|X_s|) < \infty \qquad \forall s$, *and*
2. $\mathbb{E}(X_{t+1}|X_1, X_2, ..., X_t) = X_t$, *or equivalently,* $\mathbb{E}(X_{t+1} - X_t|X_1, X_2, ..., X_t) = 0$.

Example 6.1 *Consider an infinitely lived consumer who enjoys consumption c_t with contemporaneous utility function $u(c_t)$. The utility function $u(\cdot)$ is continuous, differentiable and concave with $u'(0) = \infty$. Future period utility is discounted according to time preference parameter β. The consumer's income y_t is drawn from distribution Y_t which has the support $(0, \infty)$. The consumer can borrow and save unlimited amounts at the risk free gross interest rate rate $R = 1/\beta$.*

The consumer brings wealth w_t into period t. Immediately prior to the realisation of the shock y_t, the consumer's problem is

$$v(w_t) = \max_c \mathbb{E}_{t-1} \sum_{j=0}^{\infty} \beta^j u(c_{t+j}),$$

subject to the budget constraint

$$w_{t+1} = R(w_t + y_t - c_t)$$

and no Ponzi condition, $\lim_{T\to\infty} w_t \geq 0$.

Notation: The operator $\mathbb{E}_t(\cdot) = \mathbb{E}(\cdot|\Omega_t)$, where $(y_1, y_2, ..., y_t) \in \Omega_t$.

Show that $u'(c)$ is a martingale.

Solution 6.1 *First, re-write the consumer's objective function using a recursive form:*

$$v(w_t) = \max_{c_t, w_{t+1}} \mathbb{E}_{t-1}[u(c_t) + \beta v(w_{t+1})]$$

Subject to the budget constraint

$$w_{t+1} = R(w_t + y_t - c_t)$$

We can re-write this as a Lagrangian,

$$v(w_t) = \max_{c_t, w_{t+1}} \mathbb{E}_{t-1}[u(c_t) + \beta v(w_{t+1}) - \lambda_t(w_{t+1} - R(w_t + y_t - c_t))]$$

The first order conditions are

$$u'(c_t) - \lambda_t R$$

$$\beta v'(w_{t+1}) - \lambda_t$$

The envelope condition is

$$v(w_t) = \mathbb{E}_{t-1}\lambda_t R.$$

Combining these conditions, we get

$$u'(c_t) = \beta R \mathbb{E}_t u'(c_{t+1}).$$

Recalling that $1/\beta = R$*,*

$$\mathbb{E}_t u'(c_{t+1}) - u'(c_t) = 0.$$

This satisfies Definition 6.2 part 1. Part 2 can also be easily verified (LTS).

What does Example 6.1 tell us about the evolution of wealth inequality?

Definition 6.3 *A discrete time stochastic process X is said to be a* submartingale *if and only if*

1. $\mathbb{E}(|X_s|) < \infty \qquad \forall s$*, and*
2. $\mathbb{E}(X_{t+1}|X_1, X_2, ..., X_t) \geq X_t$.

The martingale hypothesis

Let's start with a simple model $\mathcal{M}$ mapping information Ω to price movements. The martingale hypothesis predicts that innovations in prices follow a submartingale, with constant expected return μ. This constant expected return can be interpreted as a discount rate, and may be adjusted for risk. The motivation for this model is that the information set Ω_t should be incorporated into prices p_t. Innovations in prices p_t in excess of the expected return μ should only result from new information revealed at time $t+1$, $\omega_{t+1} \notin \Omega_t$.

Our model $\mathcal{M}$ can be described as follows:

$$\mathbb{E}[p_{t+1}|\Omega_t] = (1+\mu)p_t, \tag{6.1}$$

where $\mu > 1$ is a constant. This can be rearranged to yield

$$\mu = \frac{\mathbb{E}[p_{t+1}|\Omega_t] - p_t}{p_t}$$

or alternatively,

$$\mathbb{E}[r_{t+1}|\Omega_t] = \mu. \tag{6.2}$$

Recall that μ is a constant, and is therefore independent of the elements of Ω_t. We can form the unconditional expectation $\mathbb{E}[r_{t+1}]$, which is equal to the conditional expectation $\mathbb{E}[r_{t+1}|\Omega_t]$:

$$\mathbb{E}[\mathbb{E}[r_{t+1}|\Omega_t]] = \mathbb{E}[r_{t+1}] = \mu.$$

It follows that

$$\mathbb{E}[r_{t+1}|r_t, r_{t-1}, ...] = \mu.$$

Let ε denote excess returns over and above the mean return μ,

$$\varepsilon_t = r_t - \mu.$$

It follows that expected excess returns are independent of past excess returns,

$$\mathbb{E}[\varepsilon_{t+1}|\varepsilon_t, \varepsilon_{t-1}, ...] = 0. \tag{6.3}$$

Now that we've described out model, the next step is to derive predictions of the model that we can test against the data. Proposition 6.1 derives a theoretical restriction on the relationship between past signals ω_t and future returns r_{t+1}.

Proposition 6.1 *The martingale hypothesis predicts that future returns have zero covariance with any signal in the information set, $\omega_t \in \Omega_t$, that is*

$$cov(r_{t+1}, \omega_t) = 0, \qquad \forall \omega_t \in \Omega_t.$$

Proof. Start from the definition of covariance,

$$\text{cov}(r_{t+1}, \omega_t) = \mathbb{E}[(r_{t+1} - \mathbb{E}[r_{t+1}])(\omega_t - \mathbb{E}[\omega_t])]$$

Using the law of iterated expectations, $\mathbb{E}[\mathbb{E}[Y|Z]] = \mathbb{E}[Y]$, we have

$$\begin{aligned}\mathbb{E}[(r_{t+1} - \mathbb{E}[r_{t+1}])(\omega_t - \mathbb{E}[\omega_t])] &= \mathbb{E}[\mathbb{E}[(r_{t+1} - \mathbb{E}[r_{t+1}])(\omega_t - \mathbb{E}[\omega_t])|\omega_t]] \\ &= \mathbb{E}[\mathbb{E}[(r_{t+1} - \mu)(\omega_t - \mu_\omega)|\omega_t]]\end{aligned}$$

Note that here, ω_t is known, so we can use the linearity of the expectation operator to re-write the Right hand side as follows:

$$\begin{aligned}\mathbb{E}[(r_{t+1} - \mathbb{E}[r_{t+1}])(\omega_t - \mathbb{E}[\omega_t])] &= \mathbb{E}[(\omega_t - \mu_\omega)\mathbb{E}[(r_{t+1} - \mu)|\omega_t]] \\ &= \mathbb{E}[(\omega_t - \mu_\omega)0|\omega_t]] \\ &= 0.\end{aligned}$$

■

Proposition 6.1 does provide a testable hypothesis. *Past* information signals should not affect *future* returns. Of course, the challenge from an empirical perspective is designing an experiment where we can gather many similar signals in order to gain statistical power.

We'll look at two examples of this. In the following section we'll use earnings announcements as signals in the publicly available information set Ω_{pub}. This will provide us a test of ([semi-strong-form efficiency] $\wedge$ [martingale hypothesis]) derived from Proposition 6.1.

First, we'll restrict the signal set to past returns (Corollary 6.1). This will gain us statistical power in identification but unfortunately it means we are working with a less powerful hypothesis. Rather than testing ([semi-strong-form efficiency] $\wedge$ [martingale hypothesis]), restricting Ω to Ω_P means that we will be testing ([weak-form efficiency] $\wedge$ [martingale hypothesis])

Corollary 6.1 *The martingale hypothesis predicts that returns have zero autocovariance for all lags, that is*

$$cov(r_t, r_{t-k}) = 0 \qquad \forall k > 0.$$

Proof left to student.

7
Hedging and the Fundamental Valuation Relationship

Much food is in the tillage of the poor:
but there is that is destroyed for want of judgment.
Proverbs 13:23.

Diversify your millions, you can live off the interest
Make every revenue stream flood, see where it took me.
Xzibit, *Everything*

Risk and return

Risk doesn't guarantee high expected returns. Should it?

Hedges Let asset A be a perfect hedge for some risky asset B. This means that there is some portfolio $W = X_A A + X_B B$ such that $\text{var}(W) = 0$. The expected payoff of portfolio W is $\mathbb{E}[W] = X_A \mathbb{E}[A] + X_B \mathbb{E}[B]$. The cost of portfolio W is $P_W = P_A X_A + P_B X_B$. The return on portfolio W is

$$\begin{aligned} 1 + r_W &= \frac{W}{P_W} \\ &= \frac{X_A A + X_B B}{P_A X_A + P_B X_B} \\ &= \frac{X_A A}{P_A X_A + P_B X_B} + \frac{X_B B}{P_A X_A + P_B X_B} \\ &= \frac{X_A P_A}{P_A X_A + P_B X_B}(1 + r_A) + \frac{X_B P_B}{P_A X_A + P_B X_B}(1 + r_B) \end{aligned}$$

The return on portfolio W is a weighted average of the returns on assets A and B. It follows that either $r_A < r_W$, $r_B < r_W$, or $r_A = r_W = r_B$. But both assets A and asset B are more risky (ie. higher variance) than portfolio W. Therefore, it cannot be the case that risky assets as measured by variance alone always earn higher returns than lower risk assets.

Insurance Insurance products are financial assets. Insurance premia purchase rights to future income, conditional upon pre-specified events such as car accidents, thefts, fires, illness, death, life[1]...

What is the expected return on insurance products? Probably less than the expected return on investment products. The insurance company has similar investment opportunities as an investment firm, but typically takes less investment risk. The insurance company also has higher administrative costs than an investment firm (someone needs to check that you have had a car accident, been robbed, had a fire, fallen ill, died, or are still alive). What's more, insurance premia are taxed in this country(!), whereas standard financial investments are typically subsidised (for example ISAs and pension contributions).

What is the variance of insurance products? High! Hopefully you don't have a car accident, you don't get robbed, you don't have a fire, in which case you won't receive anything from your insurance company (it is more difficult to escape illness death and life, but the insurance payoffs are still uncertain).

So if insurance has a lower expected return than standard investment products, and carries high risk, why do people buy it?

[1] An annuity is insurance against life---you get a higher payoff the longer you live.

The short answer is that insurance products, while themselves risky and low return, reduce the total risks that you are exposed to. They provide a hedge for your other risky activities, like driving. Insurance is risky, but it reduces your total risk. You pay for the privilege through low expected returns.

Equity In Lecture 1, we saw that returns to equity are much more volatile than returns to bonds and short term deposits. We also saw that this risk typically comes with reward. Equities have historically outperformed bonds over long time horizons. Why do shareholders demand high returns in return for share price risk, when other risks do not necessarily offer high returns?

Inidividual equities have high risk. Some of this risk, but not all of this risk, can be managed through diversification. Even with diversification across individual stocks, the wider stock market provides a risky return, rising and falling. How should this risk be priced?

Figure 7.1 presents the annual consumption growth and annual returns to equity for the United States. Equity returns are not just risky, but they move with consumption ($\rho = 0.39$). When consumption growth is high, returns to shareholders are high; when consumption growth is low, returns to shareholders are low. Does this correlation between consumption and returns to equity offer clues to why shareholders earn so much higher returns than bondholders and depositors?

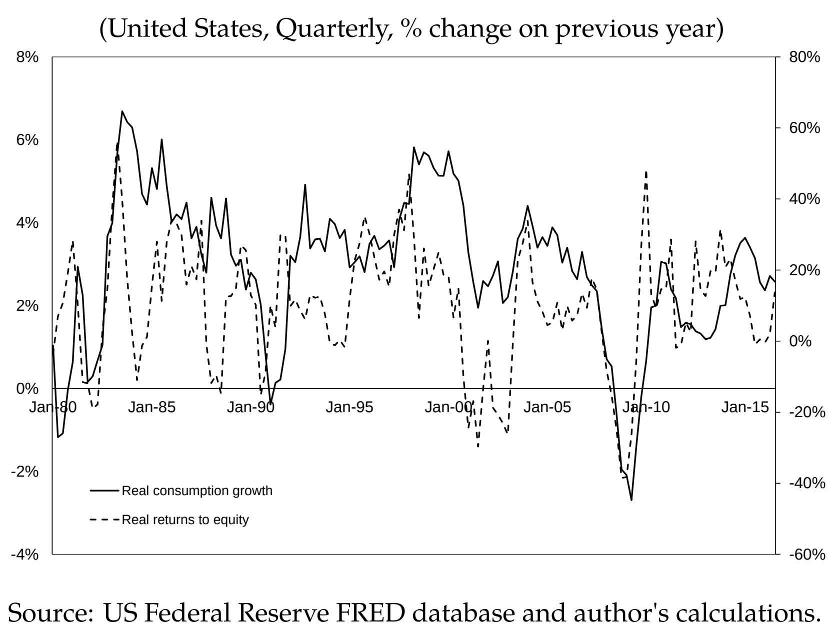


Figure 7.1: Consumption growth (LHS) and returns to equity (RHS)

The Fundamental Valuation Relationship

Consider an investor with wealth w, ready to invest in assets x_j. The future states of the world are indexed by i. The investor's consumption c^i is a random variable, as is the return for each asset, r_j^i. How should the investor allocate their initial wealth across assets x^j, and what does this tell us about risk and return?

Remember that asset pricing is about *beliefs*, *preferences* and *arbitrage*. In the FVR, $\mathbb{E}$ captures the investor's beliefs, while $u'(c)$ and $v'(w)$ capture the investors' preferences. Arbitrage-free pricing is implied as a consequence.

Theorem 7.1 *The Fundamental Valuation Relationship: Under the optimal portfolio allocation, returns satisfy the following condition*

$$1 = \mathbb{E}\left[\frac{u'(c)}{v'(w)}(1 + r_j)\right] \tag{7.1}$$

Table 7.1: Notation for Theorem 7.1

Notation for Theorem 7.1 (RV indicates random variables)			
i	Indexes states of the world.	π^i	The probability of state i occurring.
w	Initial wealth of the investor.	c	Final consumption (RV).
$v(w)$	Value function.	$u(c)$	Contemporaneous utility function (RV).
$v'(w)$	The *marginal value* of wealth.	$u'(c)$	The *marginal utility* of consumption (RV).
j	Indexes assets.	x_j	Asset j.
p_j	The price of asset j.	z_j	The payoff of asset j (RV).
r_j	The return on asset j (RV).	λ, μ	Lagrange Multipliers (λ^i is a RV).

Proof. The investor's problem is the following

$$v(w) = \max_{c^i, x_j} \mathbb{E}[u(c^i)]$$

Subject to the constraints

$$c^i \leq \sum_j z_j^i x_j \qquad \forall i,$$

and

$$w \geq \sum_j p_j x_j.$$

We can re-write this as a Lagrangian,[2]

[2] Exercise: Why is this a Lagrangian and not a Kuhn-Tucker problem?

$$\mathcal{L} = \mathbb{E}\left[u(c^i) - \lambda^i \left(c^i - \sum_j z_j^i x_j\right)\right] + \mu \left[w - \sum_j p_j x_j\right],$$

where $\lambda^1, \lambda^2, ..., \lambda^n, \mu$ are Lagrange multipliers. The Lagrange multiplier λ^i can be interpreted as the shadow value of income upon the realisation of state i. The Lagrange multiplier μ can be interpreted as the shadow value of wealth, prior to the realisation of the state i.

Lets re-write this in terms of the probabilities π^i of individual states i,

$$\mathcal{L} = \sum_i \pi^i \left[u(c^i) - \lambda^i \left(c^i - \sum_j z_j^i x_j \right) \right] + \mu \left[w - \sum_j p_j x_j \right],$$

The first order necessary conditions of this problem are:

Remember, first order conditions tell us about trade-offs. The marginal benefit of buying more of asset x_j is the increase in income the investor gets from the payoff of x_j, $\sum_i [\pi^i \lambda^i z_j^i]$. The marginal cost is μp_j. The Lagrange multipliers scale these costs and benefits to where they have the biggest effect on utility.

$$\mathcal{L}_{c^i}: \quad 0 = \pi^i [u'(c^i) - \lambda^i] \tag{7.2}$$

$$\mathcal{L}_{x_j}: \quad 0 = \sum_i [\pi^i \lambda^i z_j^i] - \mu p_j \tag{7.3}$$

$$\mathcal{L}_{\lambda^i}: \quad 0 = \pi^i \left(c^i - \sum_j z_j^i x_j \right) \tag{7.4}$$

$$\mathcal{L}_{\mu}: \quad 0 = w - \sum_j p_j x_j \tag{7.5}$$

The envelope condition for w will also be helpful here:

$$v'(w) = \mu. \tag{7.6}$$

Let's focus on $\mathcal{L}_{x_j}$, which is a statement about the optimal allocation of wealth into assets x_j. We can re-write (7.3) as follows:

We get envelope conditions by differentiating the investor's problem with respect to parameters---things the investor cannot choose, in this case their initial wealth. Envelope conditions tell us how much better or worse off the investor would be in response to an increase in that parameter.

$$\mathbb{E}[\lambda z_j] = \mu p_j$$

$$\mu = \mathbb{E}\left(\lambda \frac{z_j}{p_j} \right)$$

Note that $\dfrac{z_j}{p_j}$ is the (stochastic) gross return on asset j, $1 + r_j$.

$$\mu = \mathbb{E}\left[\lambda (1 + r_j) \right]$$

Now we have our optimal portfolio allocation in terms of the Lagrange multipliers. We can use Equations 7.2 and 7.6 to re-write as follows:

$$v'(w) = \mathbb{E}\left[u'(c)(1 + r_j) \right]$$

Now, divide through by $v'(w)$ to complete the proof:

$$1 = \mathbb{E}\left[\frac{u'(c)}{v'(w)} (1 + r_j) \right]$$

■

The FVR and marginal rates of substitution and of transformation

We can deconstruct the Fundamental Valuation Relationship to gain intuition and understanding. For each state i, the marginal rate of substitution from initial wealth to state i consumption is

$$\mathrm{MRS}_{w}^{c^i} = \frac{u'(c^i)}{v'(w)}.$$

The marginal rate of transformation from initial wealth to state i consumption for each asset j is

$$\mathrm{MRT}_{w}^{c^i} = \frac{1}{1 + r_j^i}.$$

We can re-write the fundamental valuation relationship as follows:

$$1 = \mathbb{E}\left[\frac{\mathrm{MRS}_{w}^{c^i}}{\mathrm{MRT}_{w}^{c^i}}\right].$$

This ties us back to allocative efficiency, the expectation of the quotient of stochastic marginal rates of substitution over marginal rates of transformation is 1. This is the stochastic analogue of the standard allocative efficiency requirement, MRS $=$ MRT. As returns and consumption are stochastic, the individual investor cannot equate their consumption marginal rates of substitution to the marginal rates of transformation offered by the available assets in each state. Loosely speaking, optimality or individual level efficiency requires that the investor minimises expected deviations between consumption marginal rates of substitution across states and the marginal rates of transformation across states offered by the available assets.

The FVR and the marginal contribution to expected utility

Another way to deconstruct the FVR is to consider the marginal contribution of each asset to expected utility. First, re-write the Fundamental Valuation Relationship (7.1) as follows:

$$v'(w) = \mathbb{E}\left[u'(c)(1 + r_j)\right].$$

Consider the terms on the right hand side. The final term, $(1 + r_j)$, is the realised return for asset j, a random variable. The first term, $u'(c)$ is the realised marginal utility of consumption, also a random variable. The product of these two terms $u'(c)(1 + r_j)$, is the realised contribution to utility per unit increase in holdings of asset j, at the margin. That is, if we invest a dollar more in asset j, then this will increase our realised utility by $u'(c)(1 + r_j)$, a random variable. The expected increase in realised utility resulting from a marginal increase in holdings of asset j is $\mathbb{E}\left[u'(c)(1+r_j)\right]$. What the FVR tells us is that marginal contribution to expected utility must be the same across all assets in our portfolio.

Let

$$\mathbb{E}\left[u'(c)(1+r_i)\right] < \mathbb{E}\left[u'(c)(1+r_j)\right]$$

for risky assets i and j. The investor would wish to reduce their holdings of asset i, for a utility loss of $\mathbb{E}\left[u'(c)(1+r_i)\right]$, and increase their holdings of asset j, for a utility gain of $\mathbb{E}\left[u'(c)(1+r_j)\right]$. This trade would leave the investor with increased expected utility.

So what?

So far, we've derived the fundamental valuation relationship, and shown that it is just a restatement of our standard, ECON 1, allocative efficiency condition. But what does this tell us about asset pricing?

Corollary 7.1 *Let r_0 be the return on a risk free asset. The return on a risky asset in any optimal allocation satisfies the following condition:*

$$r_0 = \mathbb{E}[r_j] + \frac{cov(u'(c), r_j)}{\mathbb{E}[u'(c)]} \tag{7.7}$$

Proof. Start with Theorem 7.1, which must hold for both the risky asset j and the risk free asset 0. For the risky asset, we have

$$1 = \mathbb{E}\left[\frac{u'(c)}{v'(w)}(1+r_j).\right] \tag{7.1}$$

For the risk free asset, we have

$$\begin{aligned}1 &= \mathbb{E}\left[\frac{u'(c)}{v'(w)}(1+r_0)\right] \\ &= (1+r_0)\mathbb{E}\left[\frac{u'(c)}{v'(w)}\right]\end{aligned}$$

Combining the two and rearranging we have

$$\begin{aligned}(1+r_0)\mathbb{E}\left[\frac{u'(c)}{v'(w)}\right] &= \mathbb{E}\left[\frac{u'(c)}{v'(w)}(1+r_j)\right] \\ (1+r_0)\mathbb{E}\left[u'(c)\right] &= \mathbb{E}\left[u'(c)(1+r_j)\right]\end{aligned}$$

$$\begin{aligned}1+r_0 &= \frac{\mathbb{E}\left[u'(c)(1+r_j)\right]}{\mathbb{E}\left[u'(c)\right]} \\ &= \frac{\mathbb{E}\left[u'(c)\right]\mathbb{E}\left[1+r_j\right] + \text{cov}\left(u'(c), (1+r_j)\right)}{\mathbb{E}\left[u'(c)\right]} \\ &= \frac{\mathbb{E}\left[u'(c)\right]\mathbb{E}\left[1+r_j\right]}{\mathbb{E}\left[u'(c)\right]} + \frac{\text{cov}\left(u'(c), (1+r_j)\right)}{\mathbb{E}\left[u'(c)\right]} \\ &= \mathbb{E}\left[1+r_j\right] + \frac{\text{cov}\left(u'(c), (1+r_j)\right)}{\mathbb{E}\left[u'(c)\right]}\end{aligned}$$

$$r_0 = \mathbb{E}\left[r_j\right] + \frac{\text{cov}\left(u'(c),(1+r_j)\right)}{\mathbb{E}\left[u'(c)\right]}$$

■

Corollary 7.1 tells us that we can relate expected asset returns to the covariance of returns and consumption marginal utility. Assets whose returns negatively covary with consumption marginal utility $(\text{cov}(u'(c), r_j) < 0)$ should yield higher expected returns than the risk free rate, $\mathbb{E}[r_j] > r_0$.

Recall that marginal utility is decreasing in consumption. So, we have that assets whose returns positively covary with consumption $(\text{cov}(c, r_j) > 0)$ should yield higher expected returns than the risk free rate, $\mathbb{E}[r_j] > r_0$.

Summary

The fundamental valuation relationship tells us that assets whose returns negatively covary with marginal utility should offer high returns. These are assets whose payoffs are high exactly when we are already doing well.

Equities have this feature. Stocks have high returns during booms, when consumption is growing quickly. Stocks offer low returns during recessions, when consumption growth is low and when we really need the money. It is not just the fact that stocks are risky that motivates their high returns, it is the fact that this risk amplifies the consumption risks we already face.

On the other hand, insurance products offer low expected returns. Insurance products have high realised returns exactly when we need the money, after an accident or when we suffer illness. Insurance, while itself risky, reduces our total risks.

These general lessons are characterised by the Fundamental Valuation Relationship, and are perhaps intuitive. The value of the Fundamental Valuation Relationship in its mathematical form is largely as a means for econometric analysis. For example, it gives us an econometric framework with which we can link equity risk premia to investors risk aversion.

At least, this is what you'll be looking at next term. There are lots of lessons we can learn from the fundamental valuation relationship. It is powerful, and, fundamental. I have a paper that shows, using the fundamental valuation relationship, how short termism can emerge among firms and banks in the wake of financial crises. This short termism prolongs and deepens recessions emerging from financial crises.

Problems for Chapter 7

Exercise 7.1 *Asset x_j has an expected return that is less than the risk free rate ($\mathbb{E}[r_j] < r_0$). Using the fundamental valuation relationship, what can you say about asset x_j?*

Exercise 7.2 *Describe the fundamental valuation relationship. Using examples, explain how the fundamental valuation relationship relates the risk and return of financial assets.*

Exercise 7.3 *(This is quite challenging!) Sam enjoys consumption c according to utility function $u(c)$, which is strictly increasing, concave and three times differentiable ($u', -u'' > 0$). Let the expectation of c be denoted by $\mu_c = \mathbb{E}[c]$.*

i *Take a second order Taylor series of marginal utility ($u'(c)$) around $c = \mu_c$. Show that marginal utility can be written as follows:*

$$u'(c) = u'(\mu_c) + u''(\mu_c)(c - \mu_c) + u'''(\mu_c)\frac{(c - \mu_c)^2}{2} + \mathcal{O}((c - \mu_c)^3).$$

ii *Show, using (i) and Corollary 7.1, that you can approximate the fundamental valuation relationship as follows for risky asset j and risk free asset 0,*

$$\mathbb{E}\left[r_j\right] \approx r_0 + R(\mu_c)cov\left(\log(c), r_j\right),$$

where $R(c)$ is the Arrow-Pratt measure of relative risk aversion, $R(c) = \frac{-cu''(c)}{u'(c)}$.

iii *Consider the equation derived in part (ii) of this question. Explain how an econometrician could estimate $R(\mu_c)$ from financial and consumption data.*

iv *Find some data and estimate $R(\mu_c)$.*

8
Mean variance analysis

Introduction

One of the main themes in applied economics and finance is a trade-off between realism and applicability. In Chapter 7, we developed the Fundamental Valuation Relationship, expressed in Corollary 7.1 as follows

$$\mathbb{E}[r_j] = r_0 - \frac{\text{cov}(u'(c), r_j)}{\mathbb{E}[u'(c)].}$$

This is a powerful asset pricing model, relying on strong but reasonable assumptions. Specifically, while the FVR does rely on the Expected Utility Hypothesis, it is general enough to incorporate some most classical models of utility and can even accommodate some aspects of behavioural finance, including Prospect Theory (**?**).

But this generality and realism comes at a cost when the Fundamental Valuation Relationship is applied in practise. Importantly, econometricians and finance practitioners do not observe the utility function $u(c)$. We don't always have the data that we need to estimate and apply the Fundamental Valuation Relationship in its general form, but if we make additional restricting assumptions, we can develop asset pricing models that are much easier to use in empirical and practical applications. The most important and commonly used simplification is mean-variance utility. This Chapter presents an introduction to mean-variance utility analysis, outlining some of the main drawbacks of this approach to financial modelling before considering some of the applications that make mean-variance utility such a powerful tool.

We start by defining what we mean by mean-variance utility:

Definition 8.1 *A utility function $u(w)$ is described as a mean-variance utility function if and only if*

$$\mathbb{E}[u(w)] = v(\mu_w, \sigma_w^2)$$

for some function v. That is, if expected utility can be defined strictly in terms of expected consumption and the variance of consumption.

Example 8.1 *The utility function $u(w) = \mu_w - \rho\sigma_w^2$, with ρ a constant, is a mean-variance utility function that is linear in both the expectation and the variance of consumption.*

Example 8.2 *Let $u(w)$ be a utility function. The second order Taylor exansion of $u(w)$ around μ_w is*

$$u(w) \approx u(\mu_w) + (w - \mu_w)u'(\mu_w) + \frac{(w - \mu_w)^2}{2}u''(\mu_w),$$

which is a mean-variance utility function. Now take the expectation of $u(w)$,

$$
\begin{aligned}
\mathbb{E}[u(w)] &\approx \mathbb{E}\left[u(\mu_w) + (w-\mu_w)u'(\mu_w) + \frac{(w-\mu_w)^2}{2}u''(\mu_w)\right] \\
&\approx \mathbb{E}[u(\mu_w)] + \mathbb{E}[(w-\mu_w)u'(\mu_w)] + \mathbb{E}\left[\frac{(w-\mu_w)^2}{2}u''(\mu_w)\right] \\
&\approx u(\mu_w) + \mathbb{E}[w-\mu_w]u'(\mu_w) + \frac{\mathbb{E}[(w-\mu_w)^2]}{2}u''(\mu_w) \\
&\approx u(\mu_w) + \frac{\sigma_w^2}{2}u''(\mu_w)
\end{aligned}
$$

Example 8.2 shows that for any single argument utility function, expected utility can be approximated to second order by a mean-variance utility function.

Some drawbacks of using mean-variance utility

State-contingent preferences

Mean-variance utility functions are single argument.[1] Utility depends on realised wealth alone, and a given level of wealth provides a given level of utility regardless of the state. For example, the utility derived from a given level of wealth may be dependent on employment circumstances and hours worked---this link between employment outcomes and preferences over wealth is standard in modern macroeconomic models.

[1] That is, they take the form $u(c)$, rather than $u(c, x_1, x_2, ...)$.

But we can also think of longer term risks that may affect the utility we derive from wealth. Climate change is an example, innovation in health care (or conversely antibiotic resistance) may be other examples of risks that directly and indirectly affect the utility we derive from wealth.

Higher moments

Mean-variance utility functions summarise risk by variance alone. Consider Figure 8.1, which plots normal and lognormal distributions sharing identical means and variances. These two distributions are clearly very different, and we would expect that two assets whose payoffs were represented by these two distributions would likely have different prices. The normal distribution has zero skewness and zero kurtosis. The lognormal distribution has positive skewness and kurtosis. The normal distribution permits negative payoffs, while the lognormal distribution does not.

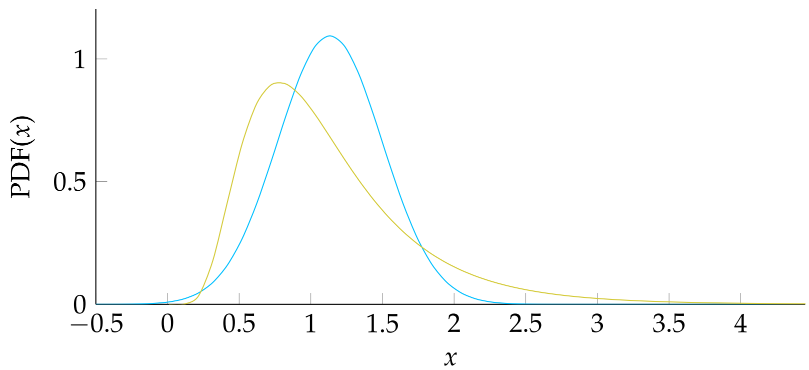


Figure 8.1: Probability density functions for distributions with equal mean and variance (normal, and lognormal).

The mean-variance utility paradox

Proposition 8.1 *For every continuous mean variance utility function $u(\mu, \sigma)$ satisfying risk aversion, there exist two assets A and B where A state dominates B, but B is preferred over A.*

A proof of this result is beyond the scope of this Module, but the following example helps illustrate the paradox.[2]

[2] A proof is provided by **?**, Th. 2.30.

Example 8.3 *Let B be an asset that returns 0 in all states, and A be an asset that returns 10 with probability 0.01, and 0 otherwise. The investor has a mean-variance criterion $v = \mu - \frac{1}{2}\sigma^2$. The investor prefers asset B over asset A.*

$$\mu_B = 0, \qquad \sigma_B^2 = 0$$

$$\mu_A = 0.99 \times 0 + 0.01 \times 10 = 0.1$$

$$\sigma_A^2 = 0.99 \times (0 - 0.1)^2 + 0.01 \times (10 - 0.1)^2 = 0.99$$

$$v_B = 0 - \frac{1}{2} \times 0 = 0$$

$$v_A = 0.1 - \frac{1}{2} \times 0.99 = -0.395$$

$v_B > v_A$, therefore the investor prefers asset B over asset A. This is a paradox, because asset A always provides at least as high a payoff as asset B, and sometimes provides a strictly higher payoff.

Applications of mean-variance utility

The costs of using mean-variance utility, described in the previous section, are high. But as we'll see in the rest of this chapter, there are important applications of the mean-variance approach that justify its use in applications in asset pricing.

Indifference curves

Let $u(w) = \mu_w - \rho\sigma_w^2$. To plot indifference curves, hold u constant at $\bar{u}$, $\mu_w = \bar{u} + \rho\sigma_w^2$. We can plot these indifference curves in the (σ_w, μ_w) space, for given utility levels $\bar{u}$, as in Figure 8.2. In this space, indifference curves are upward sloping and convex, with utility increasing in increases in μ_w and decreases in σ_w.

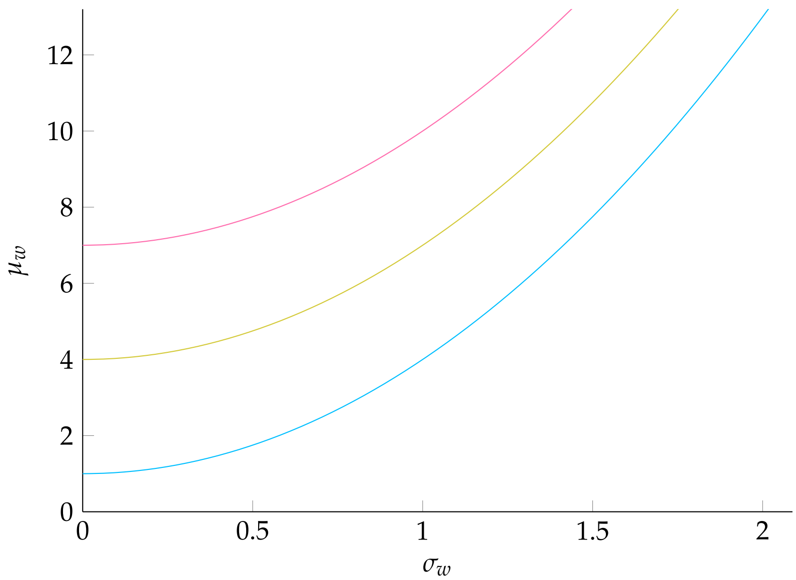


Figure 8.2: Indifference curves ($\bar{u} = \mu_w - \rho\sigma_w^2, \bar{u} = 1, 4, 7, \rho = 3$)

Varying risk aversion

Lets consider what happens when we vary ρ, the parameter that governs aversion to risk for this form of preferences. Figure 8.3 plots three different indifference curves for investors with varying risk aversion. For investors who are less tolerant to risk (higher ρ), the indifference curves become steeper than for those investors who are more tolerant to risk.

Portfolio frontiers

We now have indifference curves in the volatility-mean return space (the (σ_w, μ_w) space). The next step is to consider the attainable portfolio frontier. This portfolio frontier corresponds to the production possiblity frontiers that you'll be familiar with from other economics courses. With some combination of assets, what are the risk, return combinations (the (σ_w, μ_w) combinations) that we can construct?

Here, we need a bit of structure. We'll assume that we start with wealth $w_0 = 1$, the assets y and z each have price 1, returns $\mu_y =$

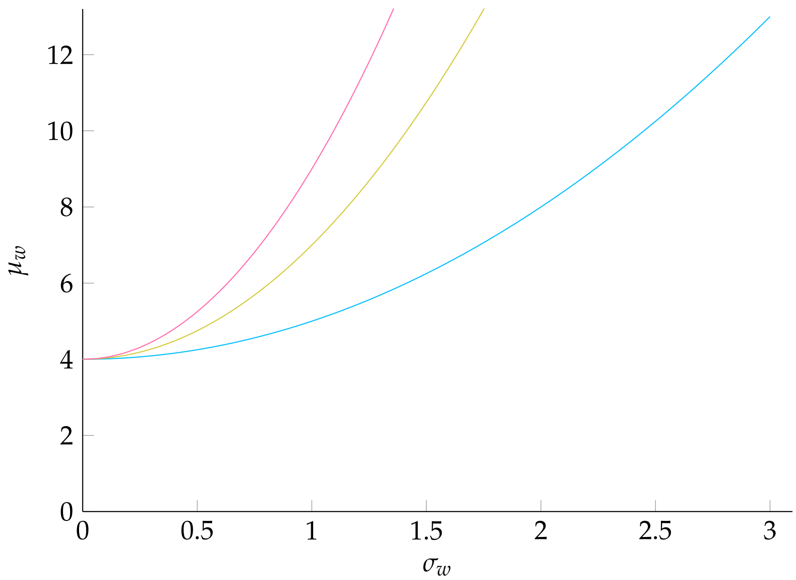


Figure 8.3: Indifference curves ($\bar{u} = \mu_w - \rho\sigma_w^2, \bar{u} = 1, \rho = 1, 3, 5$)

3, $\mu_z = 7$, volatilities $\sigma_y = 1, \sigma_z = 1.5$, covariance $\sigma_{yz} = 0.4$.

Let x_y be the units of asset y purchased and x_z be the units of asset z purchased. To construct the portfolio frontier, recall that

$$\mu_w = x_y \mu_y + x_z \mu_z.$$

We also know that

$$\sigma_w^2 = x_y^2 \sigma_y^2 + x_z^2 \sigma_z^2 + 2 x_y x_z \sigma_{yz},$$

but we need to convert this into a standard deviation,

$$\sigma_w = \sqrt{x_y^2 \sigma_y^2 + x_z^2 \sigma_z^2 + 2 x_y x_z \sigma_{yz}}.$$

The portfolios we can construct with this information are presented in Figure 8.4. The labels y and z mark the points where the portfolio holds only the asset y,z respectively. Interior points on the schedule indicate portfolios that hold both assets. The curvature of the schedule indicates the importance of diversification for reducing the volatility of portfolio returns.

Varying covariance

We can get some intuition about the shape of the portfolio frontier by varying the covariance between the two assets. Holding the rest of the parameters from Figure 8.4 constant, Figure 8.5 varies the covariance between the two assets. When $\sigma_{y,z} = -\sigma_z \sigma_y$ (or $\rho_{y,z} = -1$) there exists a portfolio with zero risk.

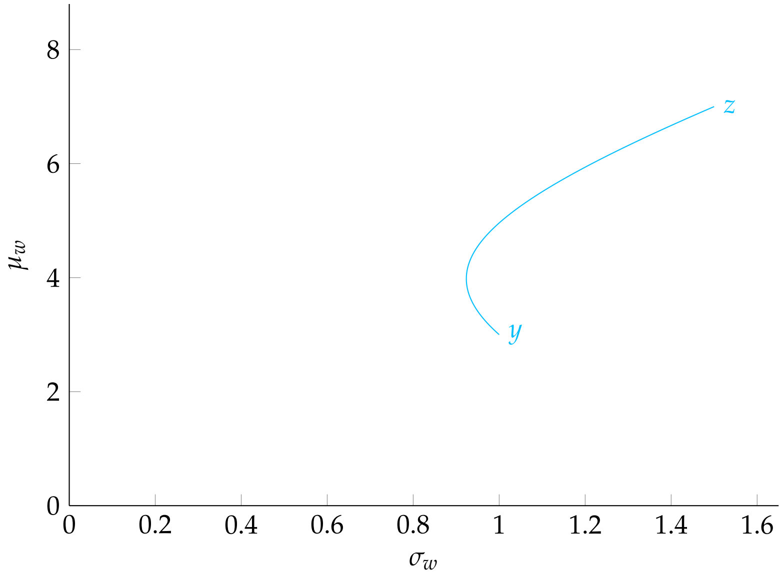


Figure 8.4: Portfolio frontier

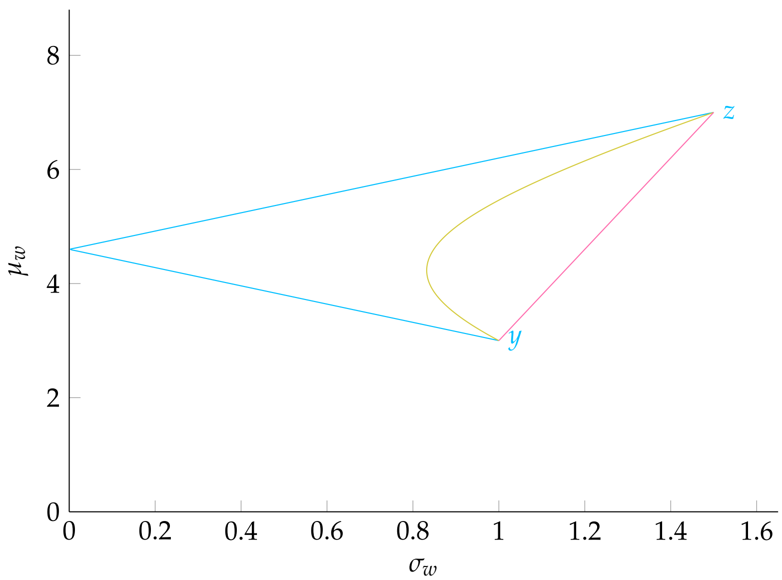


Figure 8.5: Portfolio frontiers ($\sigma_{y,z} = -\sigma_z\sigma_y, 0, \sigma_z\sigma_y$)

Optimal portfolios

Now, let's solve for our optimal portfolios. These optimal portfolios will be such that the portfolio frontier and the investor's indifference curves share tangencies.

Example 8.4 *Let z be a risky asset with price p_z, expected payoff μ_z and standard deviation σ_z. y is also a risky asset, with price p_y expected payoff μ_y and standard deviation σ_y. Both assets have price 1. Sam is an investor with* *initial wealth of w_0 and can allocate this wealth across assets x and y. Sam values expected utility according to $u(w) = \mu_w - \rho\sigma_w^2$.*

How should Sam's initial wealth be allocated across the two assets?

Solution in the z, y space

Let x_z be the units of asset z purchased, the units of asset y purchased is x_y.

It follows that,

$$\mu_w = x_z\mu_z + x_y\mu_y$$

$$\sigma_w^2 = x_z^2\sigma_z^2 + x_y^2\sigma_y^2 + 2x_z x_y \sigma_{z,y}$$

We can re-write Sam's expected utility as follows:

$$u(w) = x_z\mu_z + x_y\mu_y - \rho[x_z^2\sigma_z^2 + x_y^2\sigma_y^2 + 2x_z x_y \sigma_{z,y}]$$

To find the optimal portfolio allocation, we maximise $u(w)$ with respect to x_z, x_y, subject to the constraint

$$x_z p_z + x_y p_y = w_0. \tag{8.1}$$

Our Lagrangian is

$$\mathcal{L} = x_z\mu_z + x_y\mu_y - \rho[x_z^2\sigma_z^2 + x_y^2\sigma_y^2 + 2x_z x_y \sigma_{z,y}] + \lambda[w_0 - x_z p_z - x_y p_y].$$

The first order necessary conditions are

$$\frac{\mathcal{L}}{dx_z}: \qquad 0 = \mu_z - \rho\left[2x_z\sigma_z^2 + 2x_y\sigma_{z,y}\right] - \lambda p_z.$$

$$\frac{\mathcal{L}}{dx_y}: \qquad 0 = \mu_y - \rho\left[2x_y\sigma_y^2 + 2x_z\sigma_{z,y}\right] - \lambda p_y.$$

Let's take a closer look at these first order conditions. Without loss of generality, the first two terms, $\mu_z - \rho\left[2x_z\sigma_z^2 + 2x_y\sigma_{z,y}\right]$, capture the marginal benefit of holding an additional unit of the asset z. [3] This marginal benefit captures both the increase in expected payoff from additional units of asset z, less the cost of additional risk accruing from holding more units of asset z. The third term, $-\lambda p_z$, captures the marginal cost, in terms of resources, of purchasing an additional unit of asset z. This comprises the price of asset z, p_z, as well as the Lagrange multiplier λ, the shadow value of wealth, which summarises the opportunity cost of wealth in this environment.

[3] ``Without loss of generality", or WOLOG, means that we can replace all the z's with y's and the following statements still hold.

We can re-write these first order conditions eliminating λ as follows

$$\frac{\mu_z - \rho\left[2x_z\sigma_z^2 + 2x_y\sigma_{z,y}\right]}{\mu_y - \rho\left[2x_y\sigma_y^2 + 2x_z\sigma_{z,y}\right]} = \frac{p_z}{p_y}.$$

Hopefully this looks familiar, the left hand side is the marginal rate of substitution between assets y and z. The right hand side is the marginal rate of transformation between assets y and z.

We can solve for x_z, x_y as follows. First, rearrange the above condition such that it is linear in x_y, x_z:

$$p_y\left(\mu_z - \rho\left[2x_z\sigma_z^2 + 2x_y\sigma_{z,y}\right]\right) = p_z\left(\mu_y - \rho\left[2x_y\sigma_y^2 + 2x_z\sigma_{z,y}\right]\right).$$

Now, recall the initial wealth constraint (8.1),

$$w_0 = x_z p_z + x_y p_y.$$

We have two conditions that are linear in two unknowns, and can solve. (Left to student)

Solution in the μ_w, σ_w space

This is a bit more tricky, but perhaps more intuitive. What we want to do is re-cast our problem into expected returns and risk, away from explicit allocation between the two assets x and y. Of course, we should get the same answer. With this method, we should be able to construct efficient frontiers in our μ_w, σ_w space, and equate tangents with our indifference curves in Figure 8.2.

From our utility function, we can obtain a marginal rate of substitution between risk and reward:

$$u(w) = \mu_w - \rho\sigma_w^2$$

$$u_{\mu_w} = 1, \qquad u_{\sigma_w} = -2\rho\sigma_w$$

$$MRS_{\sigma_w,\mu_w} = \frac{u_{\mu_w}}{u_{\sigma_w}} = -\frac{1}{2\rho\sigma_w}$$

This is quite straightforward, at the margin we'd be willing to accept $2\rho\sigma_w$ units of risk for an additional unit of expected wealth. When ρ is larger, or σ_w is larger, our tolerance for further risk falls, and we need more compensation in terms of expected consumption to compensate for further increases in risk.

So far so nice and intuitive! But now we need a marginal rate of transformation from risk to reward.

Let x_z be the units of asset z purchased, the units of asset y purchased is x_y. The expected payoff of portfolio is

$$\mu_w = x_z\mu_z + x_y\mu_y \tag{8.2}$$

The variance of the portfolio is

$$\sigma_w^2 = x_z^2\sigma_z^2 + x_y^2\sigma_y^2 + 2x_z x_y\sigma_{z,y} \tag{8.3}$$

The budget constraint is

$$x_z p_z + x_y p_y = w_0. \tag{8.4}$$

What we want, is $\frac{d\mu_w}{d\sigma_w}$, holding initial wealth w_0 constant and varying x_z, x_y.

There are a few ways to do this, including to directly eliminate x_z, x_y from the above conditions and then take the derivative $\frac{d\mu_w}{d\sigma_w}$ directly. In my view, that's a bit messy, and prone to error. It is hard to keep intuition when you have long lines of algebra. So I suggest using total derivatives and the chain rule liberally.

Left to student! (Don't spend much time on this, it is useful to think about different ways to solve the same problem, that highlight different aspects of the problem. But, from here on in the algebra is pretty complicated and not hugely helpful to understanding).

Graphical representation of optimal portfolio allocation

Figure 8.6 provides a graphical representation of optimal portfolio allocation, retaining the same parameterisation from Figures 8.2 and 8.4. The optimal allocation is where the tangent of the indifference curve I meets the tangent of the portfolio frontier yz. This is where the marginal rate of substitution from risk to return equates the marginal rate of transformation from risk to return.

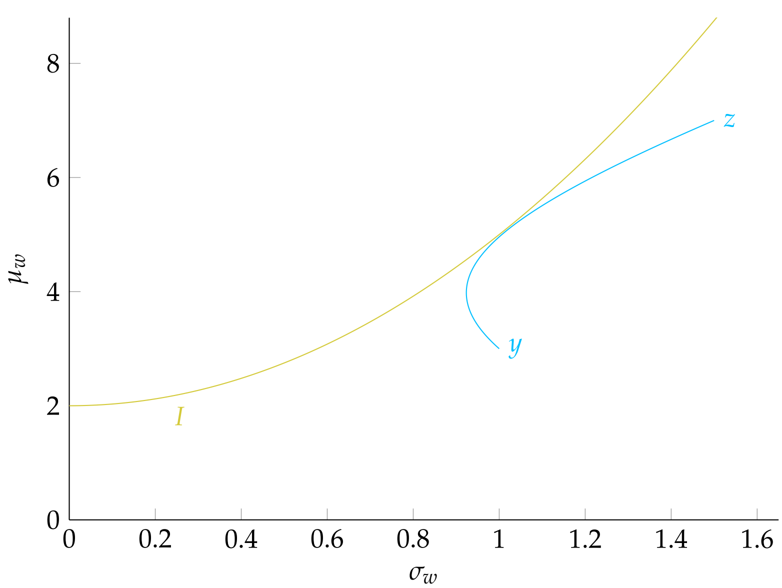


Figure 8.6: Optimal portfolio allocation

Mean-variance utility and the Fundamental Valuation Relationship

Consider an agent with all of their wealth invested in portfolio P. That is, their final consumption is given by

$$c = (1 + r_P)w,$$

where w is initial wealth and $(1 + r_P)$ is the gross return on the agent's portfolio. The agent optimises their portfolio allocation decisions to maximise a mean variance criterion

$$u(c) = \mu_c - \rho\sigma_c^2.$$

Proposition 8.2 *The risk premium of any individual asset j held by the agent can be expressed as follows:*

$$\frac{\mu_j - r_0}{\sigma_{P,j}/\sigma_P} = \frac{\mu_P - r_0}{\sigma_P} \tag{8.5}$$

Proof. Marginal utility can be expressed as

$$u'(c) = 1 - 2\rho c.$$

In terms of portfolio returns, we have

$$u'(c) = 1 - 2\rho(1 + r_P)w. \tag{8.6}$$

By Corollary 7.1, the portfolio optimality condition for asset j can be described as follows

$$\begin{aligned}
\mu_j &= r_0 - \frac{\operatorname{cov}(u'(c), r_j)}{\mathbb{E}[u'(c)]} \\
&= r_0 - \frac{\operatorname{cov}(1 - 2\rho(1 + r_P)w, r_j)}{\mathbb{E}[u'(c)]} \qquad \text{by (8.6)} \\
&= r_0 - \frac{-2\rho w \operatorname{cov}(r_P, r_j)}{\mathbb{E}[u'(c)]} \\
&= r_0 + \frac{2\rho w \operatorname{cov}(r_P, r_j)}{\mathbb{E}[u'(c)]} \\
\mu_j - r_0 &= \frac{2\rho w}{\mathbb{E}[u'(c)]} \operatorname{cov}(r_P, r_j) \qquad (8.7)
\end{aligned}$$

The risk premium , $\mu_j - r_0$, is proportional to the covariance of the asset's returns with the portfolio, $\operatorname{cov}(r_P, r_j)$.

Equation 8.7 holds for all assets, including portfolio P. So we have (using $\sigma_{j,P} = \operatorname{cov}(r_j, r_P)$),

$$\mu_j - r_0 = \frac{2\rho w}{\mathbb{E}[u'(c)]} \sigma_{P,j} \tag{8.8}$$

$$\mu_P - r_0 = \frac{2\rho w}{\mathbb{E}[u'(c)]} \sigma_P^2 \tag{8.9}$$

We wish to cancel out the term $\frac{2\rho w}{\mathbb{E}[u'(c)]}$ and find an expression simply in terms of risk premia, covariance and variance terms. One option is to divide 8.8 by 8.9:

$$\frac{\mu_j - r_0}{\mu_P - r_0} = \frac{\sigma_{P,j}}{\sigma_P^2}$$

The traditional way to write down this relationship is to express the risk premia of asset j relative to the risk premia of the portfolio P. Multiplying both sides by $(\mu_P - r_0)\frac{\sigma_P}{\sigma_{P,j}}$ we have

$$\frac{\mu_j - r_0}{\sigma_{P,j}/\sigma_P} = \frac{\mu_P - r_0}{\sigma_P}$$

■

The Sharpe Ratio

Definition 8.2 *The* Sharpe ratio *of asset j is defined as*

$$s_j = \frac{\mu_j - r_0}{\sigma_j}$$

Corollary 8.1 *Assume the mean-variance FVR holds for individual asset j and portfolio P (Proposition 8.2). The Sharpe ratio of asset j is less than or equal to that of portfolio P.*

Proof. Start with the FVR relationship,

$$\frac{\mu_j - r_0}{\sigma_{P,j}/\sigma_P} = \frac{\mu_P - r_0}{\sigma_P}.$$

The fraction $\sigma_{P,j}/\sigma_P$ is equal to $\text{corr}(P,j)\sigma_j$ by Definition 1.6. So, we can write

$$\frac{\mu_j - r_0}{\text{corr}(P,j)\sigma_j} = \frac{\mu_P - r_0}{\sigma_P}.$$

Multiplying both sides by $\text{corr}(P,j)$ we have

$$\frac{\mu_j - r_0}{\sigma_j} = \text{corr}(P,j)\frac{\mu_P - r_0}{\sigma_P}.$$

The correlation $\text{corr}(P,j)$ is within the range [0,1], therefore we have

$$\frac{\mu_j - r_0}{\sigma_j} \leq \frac{\mu_P - r_0}{\sigma_P}.$$

■

The Sharpe ratio provides us with a simple and powerful measure of risk and reward that we will come back to in future lectures and problem sets. The fact that simple relationships like Proposition 8.2 and Corollary 1.6 are testament to the power of the Fundamental Valuation Relationship, which provides researchers with a wide range of predictions to help understand asset pricing and to help test hypotheses.

Within any optimal portfolio, Corollary 8.1 states that the Sharpe ratio for any individual asset is less than the Sharpe ratio for the portfolio as a whole. It may still be worthwhile to hold asset j regardless; including asset j in our portfolio may reduce the total risk of our portfolio (that is, it may lower s_P) even if asset j itself has high variance. This hints at the value of diversification that we will return to in the next chapter.

Application: Business Cycle Welfare Costs

In a provocative article, **?** sought to calculate the welfare costs of business cycles and compare these costs to the benefits of higher long run growth. Lucas showed, using the following methodology, that the welfare costs of business cycles were quite modest, similar to very small increases in the level of output. Further, Lucas argued, these modest gains provided an unobtainable upper bound on the possible welfare gains from further stabilisation---it is unlikely that business cycles could or should ever be completely eliminated. Lucas suggested that perhaps, macroeconomists need to refocus their priorities away from business cycle stabilisation and toward growth.

Let $u(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$. The parameter γ corresponds to Arrow-Pratt Relative Risk Aversion. This type of utility function is referred to as constant relative risk aversion.

Using Example 8.2, we can approximate $\mathbb{E}[u(c)]$ by the following mean-variance utility function, $v(\mu_c, \sigma_c^2) = u(\mu_c) + \dfrac{\sigma_c^2}{2} u''(\mu_c)$. In this case, we have

$$u'(c) = c^{-\gamma},$$
$$u''(c) = -\gamma c^{-\gamma-1} = -\gamma \frac{u'(c)}{c}.$$

We can re-write $v(\mu_c, \sigma_c^2)$ as follows:

$$\begin{aligned}
v(\mu_c, \sigma_c^2) &= u(\mu_c) + \frac{\sigma_c^2}{2} u''(\mu_c) \\
&= u(\mu_c) + \frac{\sigma_c^2}{2} \left(\frac{-\gamma u'(\mu_c)}{\mu_c} \right) \\
&= u(\mu_c) - \frac{\gamma}{2} \left(\frac{u'(\mu_c)}{\mu_c} \right) \sigma_c^2
\end{aligned}$$

Let $\hat{c}$ be the level of consumption that provides the same utility as μ_c, σ_c^2 when risk is zero. In other words, $v(\hat{c}, 0) = v(\mu_c, \sigma_c^2)$. $\hat{c}$ is the certainty equivalent consumption bundle. The ratio $\frac{\hat{c} - \mu_c}{\mu_c}$ is the consumption equivalent cost of business cycles, the proportion of expected consumption that would be sacrificed by households in order to eliminate business cycle risk.

$$\begin{aligned} 0 &= v(\hat{c}, 0) - v(\mu_c, \sigma_c^2) \\ &= u(\hat{c}) - u(\mu_c) + \frac{\gamma}{2} \left(\frac{u'(\mu_c)}{\mu_c} \right) \sigma_c^2 \end{aligned}$$

Take a first order Taylor series approximation of $u(\hat{c}) - u(\mu_c)$ around μ_c:

$$u(\hat{c}) - u(\mu_c) = (\hat{c} - \mu_c) u'(\mu_c)$$

Now, substitute in this approximation:

$$(\hat{c} - \mu_c) u'(\mu_c) = -\frac{\gamma}{2} \left(\frac{u'(\mu_c)}{\mu_c} \right) \sigma_c^2$$

Marginal utility $u'(\mu_c)$ appears on both sides. Also, we want to solve for $\frac{\hat{c} - \mu_c}{\mu_c}$. So, divide both sides by $u'(\mu_c)/\mu_c$:

$$\frac{\hat{c} - \mu_c}{\mu_c} = -\frac{\gamma}{2} \left(\frac{\sigma_c}{\mu_c} \right)^2$$

For the US, we can obtain the following values: $\left(\frac{\sigma_c}{\mu_c} \right)^2 = 0.001538$. Standard values of γ range from 0.5 to 4, where economists tend to find lower values in experimental settings, and larger values in macroeconomic settings.

Assuming that $\gamma = 0.5$, we have $\frac{\hat{c} - \mu_c}{\mu_c} = -0.038\%$. Households would be willing to give up 0.038% of their average consumption to eliminate business cycles. Average consumption in the US is about USD \$39 000 per year, so this corresponds to USD \$15 per year. If on the other hand we assumed $\gamma = 4$, households would be willing to pay USD \$120 per year per person to eliminate business cycle fluctuations.

These calculations suggest that the elimination of business cycles could at best result in a welfare gain equivalent to a permanent increase in GDP of between 0.038% ($\gamma = 0.5$) and 0.31% ($\gamma = 4$). This is very modest in terms of long run economic growth.

Problems for Chapter 8

Exercise 8.1 *Do you think economists should adjust their priorities away from macroeconomic stabilisation and towards growth?*

Exercise 8.2 *What, if anything, does the Business Cycle Welfare Costs Application in Section 8 tell you about the usefulness of mean-variance analysis? Does this application highlight any of the weaknesses of mean-variance analysis? (for example, but not limited to, those described in Section 8)*

Coding exercise for Chapter 8

Coding Exercise 8.1 *By the end of this exercise, you should have*

1. *estimated the moments of the historical monthly returns of the Total Return Russell 3000 Index (you are free to edit the index and the periodicity),*

2. *compared the distribution of historical returns to a normal distribution with the same mean and variance,*

3. *calculated the risk free equivalent rate that would offer an investor the same utility as the stock market,*

4. *and compared this risk free equivalent rate with one derived from mean-variance approximated utility, as well as higher order approximations that take into account skewness and kurtosis.*

```
#-------------------------------------------------------------------------------
# Preamble
#-------------------------------------------------------------------------------

# Load packages

using FredData
using UnicodePlots
using DataArrays
using DataFrames
using Distributions

# Assign Fred API key to access https://fred.stlouisfed.org

f = Fred("691f7fe81e035ddd29be13594f6025d6")

# If you plan to use FRED a lot, get your own API key
# https://research.stlouisfed.org/docs/api/fred/

#-------------------------------------------------------------------------------
# Download and manipulate data
#-------------------------------------------------------------------------------

RussellTR = DataFrame()
try
  RussellTR = get_data(f,"RU3000TR").df[:,3:4]  # Download data
  writetable("RussellTR.csv",RussellTR)
catch
  RussellTR        = readtable("RussellTR.csv")
  df               = Dates.DateFormat("yyyy-mm-dd")
  RussellTR[:date] = Date(RussellTR[:date],df)
end

RussellTR      = RussellTR[RussellTR[:value].>0,:] # Remove NaN values
RussellTR[:RM] = 0.0                               # Initialise return column

# Generate monthly returns series
for i in 1:size(RussellTR)[1]
  try
    RussellTR[:RM][i] = (
        log(RussellTR[:value][i])
        - log(RussellTR[(RussellTR[:date].==
                         RussellTR[:date][i] - Dates.Month(1)),:][:value][1])
                        )
  catch
    RussellTR[:RM][i] = NA                     # Exception handling
  end
end

Returns = RussellTR[~isna(RussellTR[:RM]),:]  # Remove NA return vales
lineplot(Returns[:date],Returns[:RM])         # Plot returns by date
```

```
# Moments

mur = sum(Returns[:RM])/(size(Returns)[1]);                # Mean returns
sdr = sqrt(sum((Returns[:RM]-mur).^2)/(size(Returns)[1])); # Standard deviation
skr = sum(((Returns[:RM]-mur)/sdr).^3)/(size(Returns)[1]); # Normalised skewness
kur = sum(((Returns[:RM]-mur)/sdr).^4)/(size(Returns)[1]); # Normalised kurtosis

# Compare with normal distribution

buckets = 100;
grid    = collect(linspace(minimum(Returns[:RM]),
                           maximum(Returns[:RM]),
                           buckets));

ndraws       = 10000000;
normal_draws = mur + sdr*randn(ndraws);

dist_ret = zeros(size(grid)[1]-1);
dist_norm = zeros(size(grid)[1]-1);
for i in 1:size(dist_ret)[1]
  dist_ret[i] = (size(Returns[((Returns[:RM].>=grid[i]) &
                               (Returns[:RM] .< grid[i+1])),:])[1]
                /size(Returns)[1]
                )
  dist_norm[i]  = (size(normal_draws[((normal_draws.>=grid[i]) &
                                      (normal_draws.< grid[i+1]))])[1]
                  /size(normal_draws)[1]
                  )
end

# Plot the distributions

plotreturns = lineplot(grid[1:size(grid)[1]-1],dist_ret,
                       title="Distribution of returns vs normal distribution");
lineplot!(plotreturns,grid[1:size(grid)[1]-1],dist_norm);

println(plotreturns)

# Zoom in on the left hand tail

plotlefttail = lineplot(grid[1:size(grid)[1]-1], dist_ret,
                        xlim=[grid[1]; grid[convert(Int,round(0.25*buckets,0))]],
                        ylim=[0;2*dist_ret[convert(Int,round(0.25*buckets,0))]],
                        title="Left tail of returns vs normal distribution");
lineplot!(plotlefttail,grid[1:size(grid)[1]-1],dist_norm);

println(plotlefttail)

#-------------------------------------------------------------------------------
# Utility calculations
#-------------------------------------------------------------------------------

# Functions

gamma = 1.5;                                           # CRRA coefficient
u(x)    = x.^(1-gamma)./(1-gamma);                     # CRRA utility function
u1(x)   = x.^(-gamma);                                 # Marginal utility, u'(x)
u2(x)   = -gamma.*x.^(-gamma-1);                       # u''(x)
u3(x)   = gamma*(1+gamma).*x.^(-gamma-2);              # u'''(x)
u4(x)   = -gamma*(1+gamma)*(2+gamma).*x.^(-gamma-3);   # u''''(x)
uInv(u) = ((1-gamma)*u)^(1/(1-gamma));                 # Inverse utility

# Expected utility

EU  = sum(u(1+Returns[:RM]))/(size(Returns)[1]); # True expected utility
EU1 = u(1+mur);                                  # First order approximation
EU2 = EU1 + (sdr^2/2)*u2(1+mur);                 # Second order approximation
EU3 = EU2 + (sdr^3*skr/6)*u3(1+mur);             # Third order approximation
EU4 = EU3 + (sdr^4*kur/24)*u4(1+mur);            # Fourth order approximation

# Consumption equivalent
# What is the risk free consumption bundle that offers the same
# expected utility as the risky consumption bundle?

CE   = uInv(EU);
```

```
CE1  = uInv(EU1);
CE2  = uInv(EU2);
CE3  = uInv(EU3);
CE4  = uInv(EU4);

# Risk free return equivalent
# What is the risk free rate that offers the same expected Utility
# as the risky return?

RE   = CE  - 1;
RE1  = CE1 - 1;
RE2  = CE2 - 1;
RE3  = CE3 - 1;
RE4  = CE4 - 1;

# How good are the low order approximations?

plotrf = lineplot([1;2;3;4;5],[RE1;RE2;RE3;RE4;RE],
                  title="Risk free equivalent return by order of approximation")

println(plotrf)
```

9
Efficient Portfolios

It's like a crapshoot in Las Vegas, except in Las Vegas the odds are with the house. As for the market, the odds are with you, because on average over the long run, the market has paid off.
Harry Markowitz

Fresh out the gate again, time to raise the stakes again
Fatten my plate again, y'all cats know we always play to win
Gang Starr, *Full Clip*

Floss a little; invest up in a mutual fund.
Busta Rhymes, *Dangerous*

Definition of efficient portfolios

Definition 9.1 *The return on risky asset i is denoted by $r_i \in \{r_1, r_2, r_3, ..., r_n\}$. A portfolio p is specified by portfolio weights $\{w_1, w_2, w_3, ..., w_n\}$, where w_i is the proportional allocation of the portfolio in asset i. Portfolio p is an* efficient portfolio *if and only if for all other portfolios $p' \neq p$, either $\mathbb{E}[r_p] > \mathbb{E}[r_{p'}]$ or $\sigma_p^2 < \sigma_{p'}^2$.*

Consider Definition 9.1. Any portfolio that is not an *efficient portfolio* cannot be the optimal portfolio for an investor with mean-variance utility. All investors with a mean-variance objective must hold an efficient portfolio as their optimal portfolio. For an individual investor, their choice of efficient portfolio is dependent on their risk tolerance.

The two asset case

Example 9.1 *Consider assets z and y from the previous lecture. Figure 9.1 marks the portfolio frontier, the schedule zy. The region az marks the efficient portfolios, the upward sloping section of the portfolio frontier. The portfolios along the region ya are inefficient. For every portfolio p' in ya, there exists a portfolio p in az such that $\mathbb{E}[r_p] > \mathbb{E}[r_{p'}]$ and $\sigma_p^2 \leq \sigma_{p'}^2$.*

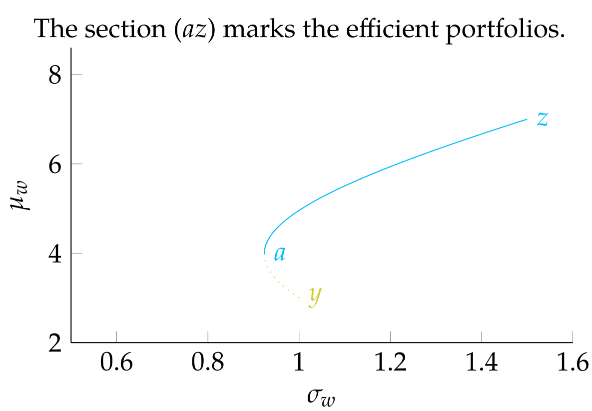


Figure 9.1: Portfolio frontier with two assets, z and y.

The Markowitz Bullet

In this section we wish to extend the above analysis to a market with many assets. The key question is the following: to what extent can diversification reduce the risk of a portfolio of assets?

First, we need a couple of new probability results.

Portfolio returns with many assets

The return on risky asset i is denoted by $r_i \in \{r_1, r_2, r_3, ..., r_n\}$. A portfolio p is specified by portfolio weights $\{w_1, w_2, w_3, ..., w_n\}$, where w_i is the proportional allocation of the portfolio in asset i. Note that $\sum_{i=1}^{n} w_i = 1$.

Property 9.1 *The return on portfolio p is*

$$r_p = w_1 r_1 + w_2 r_2 + w_3 r_3 + ... + w_n r_n$$

$$r_p = \sum_{i=1}^{n} w_i r_i$$

Property 9.2 *The variance of the return on portfolio p is*

$$var(r_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j cov(r_i, r_j)$$

Proof. By definition, we have

$$\text{var}(r_p) = \mathbb{E}[(r_p - \mu_p)^2]$$

where $\mu_p = \mathbb{E}[r_p]$. Using property 9.1 we have

$$\begin{aligned}
\text{var}(r_p) &= \mathbb{E}\left[\left(\sum_{j=1}^{n} w_j(r_j - \mu_j)\right)^2\right] \\
&= \mathbb{E}\left[w_1(r_1 - \mu_1)\left(\sum_{j=1}^{n} w_j(r_j - \mu_j)\right) + ... + w_n(r_n - \mu_n)\left(\sum_{j=1}^{n} w_j(r_j - \mu_j)\right)\right] \\
&= \mathbb{E}\left[\left(\sum_{j=1}^{n} w_1 w_j (r_1 - \mu_1)(r_j - \mu_j)\right) + ... + \left(\sum_{j=1}^{n} w_n w_j (r_n - \mu_n)(r_j - \mu_j)\right)\right] \\
&= \sum_{j=1}^{n} w_1 w_j \mathbb{E}[(r_1 - \mu_1)(r_j - \mu_j)] + ... + \sum_{j=1}^{n} w_n w_j \mathbb{E}[(r_n - \mu_n)(r_j - \mu_j)] \\
&= \sum_{j=1}^{n} w_1 w_j \text{cov}(r_1, r_j) + ... + \sum_{j=1}^{n} w_n w_j \text{cov}(r_n, r_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \text{cov}(r_i, r_j)
\end{aligned}$$

■

Three asset portfolios

What happens when we have more than two assets? Figure 9.2 constructs a sample of three asset portfolios, indicated by black dotted lines. The two asset portfolios are indicated by the black solid lines.

Here, rather than having a schedule of possible volatility-return combinations, we have an area of possible volatility-return combinations. Of this area, the left boundary traces out the *Markowitz Bullet*, the set of portfolios with minimum variance for their given mean return. The upward sloping part of the Markowitz Bullet traces the schedule of efficient portfolios.

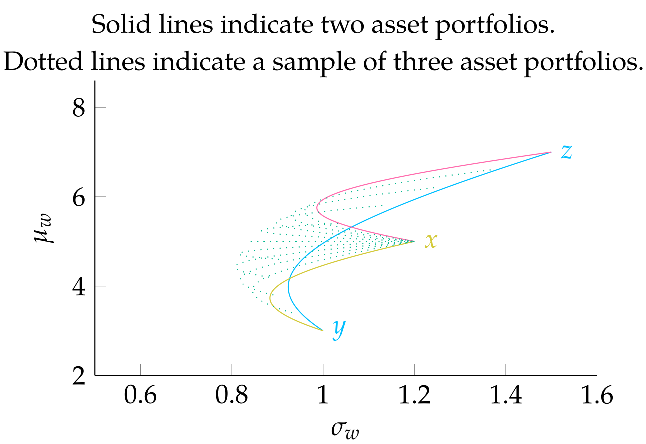


Figure 9.2: Feasible portfolios with three assets, x, y and z.

Figure 9.3 represents the Markowitz bullet for a market with many risky assets. The Markowitz bullet represents the portfolio frontier, the lowest risk portfolio for any given level of expected returns. The upward sloping section of the Markowitz Bullet, in blue, is the schedule of efficient portfolios. The downward sloping section of the Markowitz Bullet, dotted in black, is the schedule of inefficient portfolios that lie on the frontier. Note that there are many more feasible but inefficient portfolios that lie on the right hand side of the Markowitz Bullet. The area to the left of the Markowitz bullet is infeasible---there is no combination of risky assets that can produce a portfolio to the left of the Markowitz Bullet by construction.

Diversification and its limits

We've seen in the two asset case how diversified portfolios can have lower risk than either of the underlying assets. If we have many assets, can we completely eliminate risk through diversification?

Proposition 9.1 *Consider a risky market with n securities, indexed by i. All assets x_i have the same expected return $\mu_i = \mu$ and their returns have the same variance $\sigma_i^2 = \sigma^2$. All assets are uncorrelated, $\sigma_{ij} = 0, \quad \forall i \neq j$. Let p be a portfolio that is equal weighted in all assets x_i. The variance of the returns on portfolio p tend to zero in the limit as n approaches infinity.*

Blue section indicates the efficient portfolios.
The dotted black section indicates the inefficient portfolios on the frontier.
Individual assets marked in red.

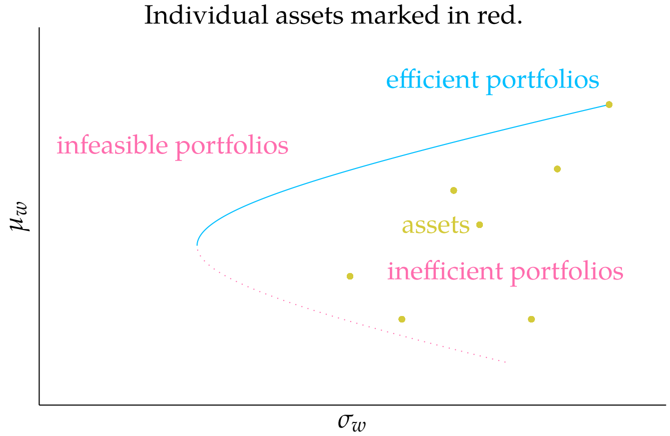


Figure 9.3: The Markowitz Bullet with many assets

Proof. Portfolio p holds equal shares of each asset. There are n assets in portfolio p, therefore the portfolio weight for any asset x_i is $w_i = 1/n$. By Property 9.2, the variance of the returns of portfolio p are

$$\begin{aligned}\operatorname{var}(r_p) &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \operatorname{cov}(r_i, r_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{n^2}\operatorname{cov}(r_i, r_j).\end{aligned}$$

Recall that $\operatorname{cov}(r_i, r_i) = \operatorname{var}(r_i)$ and that all assets are uncorrelated (if $i \neq j$, then $\operatorname{cov}(r_i, r_j) = 0$).

$$\begin{aligned}\operatorname{var}(r_p) &= \sum_{i=1}^{n} \frac{1}{n^2}\operatorname{var}(r_i) + \sum_{i=1}^{n}\sum_{j\neq i} \frac{1}{n^2}\operatorname{cov}(r_i, r_j) \\ &= \sum_{i=1}^{n} \frac{1}{n^2}\sigma_i^2 + 0 \\ &= n\frac{1}{n^2}\sigma^2 \\ &= \frac{1}{n}\sigma^2.\end{aligned}$$

This expression is always strictly positive and is decreasing in the number of assets in the portfolio, n. The question asks us to consider what happens as the number of assets n approaches infinity. To do this we need to take the mathematical limit of the above expression as

$n \to \infty$:

$$\lim_{n\to\infty} \text{var}(r_p) = \lim_{n\to\infty} \frac{1}{n}\sigma^2$$
$$= 0.$$

■

Proposition 9.1 shows that when a market consists of many uncorrelated risky assets, we can construct diversified portfolios with very little risk. It might be helpful to think about what the Markowitz Bullet looks like for a market with infinitely many uncorrelated assets as in Proposition 9.1. We've shown that it is possible in such a market to construct a riskless asset as a portfolio of risky assets. This means that the Markoqitz Bullet for this market touches the vertical axis.

In practise, individual stocks are typically positively correlated. Shocks that affect the value of Apple shares also affect the value of Microsoft shares. During recessions most stocks fall in value and during booms most stocks rise. Proposition 9.1 relied on all assets being uncorrelated, but this simplifying assumption is too strong for our purposes.. Proposition 9.2 considers a market with many correlated assets. Here there are limits to diversification. As the number of assets increases, portfolio risk converges to a strictly positive constant.

Proposition 9.2 *Consider a risky market with n securities, indexed by i. All assets x_i have the same expected return $\mu_i = \mu$ and their returns have the same variance $\sigma_i^2 = \sigma^2$. All assets are correlated with $\sigma_{ij} = \rho\sigma_i\sigma_j = \rho\sigma^2, \quad \forall i \neq j$, where $\rho > 0$. Let p be a portfolio that is equal weighted in all assets x_i. The variance of the returns on portfolio p tend to $\rho\sigma^2$ in the limit as n approaches infinity.*

Proof. Portfolio p holds equal shares of each asset. Therefore the portfolio weight in asset x_i is $w_i = 1/n$. By Property 9.2, the variance of the returns of portfolio p are

$$\text{var}(r_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \text{cov}(r_i, r_j)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{n^2}\text{cov}(r_i, r_j)$$

Recall that $\text{cov}(r_i, r_j) = \rho_{r_i,r_j}\sigma_i\sigma_j$ (by Definition 1.6).

$$\begin{aligned}
\text{var}(r_p) &= \sum_{i=1}^{n} \frac{1}{n^2}\text{var}(r_i) + \sum_{i=1}^{n}\sum_{j\neq i} \frac{1}{n^2}\rho_{r_i,r_j}\sigma_i\sigma_j \\
&= \sum_{i=1}^{n} \frac{1}{n^2}\sigma^2 + \sum_{i=1}^{n}\sum_{j\neq i} \frac{1}{n^2}\rho\sigma^2 \\
&= n\frac{1}{n^2}\sigma^2 + n(n-1)\frac{1}{n^2}\rho\sigma^2 \\
&= \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2
\end{aligned}$$

Again, we want to think about what happens when the market has many assets, so we take the limit as $n \to \infty$,

$$\begin{aligned}
\lim_{n\to\infty} \text{var}(r_p) &= \lim_{n\to\infty} \left[\frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2\right] \\
&= \rho\sigma^2.
\end{aligned}$$

■

Derivation of efficient portfolios

Let p be an efficient portfolio with mean expected return μ_p. To solve for p, we must find the portfolio that returns μ_p in expectation with the lowest possible variance. In other words, we must solve

$$\min_{w} \text{var}(r_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \text{cov}(r_i, r_j)$$

subject to the constraints that the portfolio weights add to 1,

$$1 = \sum_{i}^{n} w_i$$

and the portfolio does indeed return μ_p in expectation,

$$\mu_p = \sum_{i}^{n} w_i \mu_i,$$

where μ_i is the expected return on asset i, $\mu_i = \mathbb{E}[r_i]$.

We can write this problem as the following Lagrangian:[1]

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \text{cov}(r_i, r_j) + \lambda\left[1 - \sum_{i}^{n} w_i\right] + \nu\left[\mu_p - \sum_{i}^{n} w_i\mu_i\right].$$

[1] Note that in the Lagrangian, we have scaled the objective function by $\frac{1}{2}$. Scaling the objective function by a positive constants has no effect on the solution, but in this case will make the algebra a little bit easier.

The first order necessary conditions are

$$\frac{\partial\mathcal{L}}{\partial w_i}: \qquad 0 = \sum_{j=1}^{n} w_j \text{cov}(r_i, r_j) - \lambda - \nu\mu_i \qquad \forall i \in 1, 2, 3, ..., n. \tag{9.1}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda}: \qquad 0 = 1 - \sum_i^n w_i \tag{9.2}$$

$$\frac{\partial \mathcal{L}}{\partial \nu}: \qquad 0 = \mu_p - \sum_i^n w_i \mu_i \tag{9.3}$$

The first order conditions (9.1), (9.2) and (9.3) give us $n + 2$ linear conditions (note that (9.1) specifies n conditions) in $n + 2$ unknowns $(w_1, w_2, ..., w_n, \lambda, \nu)$. We do need some matrix algebra to solve for the optimal portfolio, which we'll leave at this stage. We can still use the first order conditions to do some interesting things.

Mutual Fund Theorems

We now have our first order conditions (9.1,9.2, and 9.3) which help us characterise the mathematical properties of mean-variance efficient portfolios in markets with many assets. In this section, we'll prove two important theorems, known as the Mutual Fund Theorems. These theorems allow us to characterise all efficient portfolios as portfolios consisting of a small number of *mutual funds*. These mutual funds are portfolios of many assets. It is useful to think of the FTSE 100 as being an example of a mutual fund in this setting.

Under the assumption of mean-variance utility, we need at most two mutual funds to construct any efficient portfolio. If we were to relax the assumption of mean-variance utility then we would need more mutual funds but the main insight would still hold true.

Theorem 9.1 *The First Mutual Fund Theorem. Consider a market with n risky assets available for investment. There exist two efficient portfolios (denoted p_1 and p_2) such that any efficient portfolio p^* can be written as a portfolio of p_1, p_2.*

This is referred to as the First Mutual Fund Theorem. We can think of portfolios p_1 and p_2 as mutual funds holding efficient portfolios of assets. Any investor maximising according to a mean variance criterion does not need to optimise their portfolio over all assets, just over the two mutual funds.

Proof.

In order to prove the theorem, we'll first need a couple of lemmas.[2] Lemma 9.1 states that any portfolio constructed from two efficient portfolios is itself efficient. Lemma 9.2 states that from any two portfolios, we can construct a third portfolio that has expected return of any value between the expected returns of the initial portfolios.

[2] A *lemma* is a building block for a larger proof. This is helpful for two reasons. First, breaking the proof into lemmas can improve readability. Second, in some circumstances it is useful to ues one lemma to help prove multiple theorems. In this case, I have used lemmas just to improve readability.

Lemma 9.1 *Let p_1 and p_2 be efficient portfolios, and p_3 be a portfolio consisting solely of portfolios p_1 and p_2 (that is, $p_3 = ap_1 + (1-a)p_2$ for some $a \in [0,1]$). Portfolio p_3 is an efficient portfolio.*

Proof. For each asset i, portfolio p_3 holds proportion $w_{3i} = aw_{1i} + (1-a)w_{2i}$ of its value in asset i. To show that portfolio p_3 efficient, we need to show that the first order conditions (9.1), (9.2) and (9.3) hold for p_3. We know that they hold for efficient portfolios p_1, p_2. Let's start with (9.1):

$$\begin{aligned}
\sum_{j=1}^{n} w_{3j}\text{cov}(r_i, r_j) - \lambda - \nu\mu_i &= \sum_{j=1}^{n}(aw_{1i} + (1-a)w_{2i})\text{cov}(r_i, r_j) - \lambda - \nu\mu_i \\
&= a\sum_{j=1}^{n} w_{1i}\text{cov}(r_i, r_j) + (1-a)\sum_{j=1}^{n} w_{2i}\text{cov}(r_i, r_j) - \lambda - \nu\mu_i \\
&= a\left[\sum_{j=1}^{n} w_{1i}\text{cov}(r_i, r_j) - \lambda - \nu\mu_i\right] \\
&\quad + (1-a)\left[\sum_{j=1}^{n} w_{2i}\text{cov}(r_i, r_j) - \lambda - \nu\mu_i\right] \\
&= a\,[0] + (1-a)\,[0] = 0.
\end{aligned}$$

We can show the same result for conditions (9.2) and (9.3) (Left to student). Portfolio p_3 satisfies the $n+2$ first order conditions, and must therefore be an efficient portfolio. ■

Lemma 9.2 *Let portfolios p_1 and p_2 have expected returns μ_1, μ_2 respectively. Let μ^* be in $[\mu_1, \mu_2]$. There exists some a such that portfolio p_3 has return $\mu_3 = \mu^*$, where portfolio is a portfolio consisting solely of portfolios 1 and 2, $p_3 = ap_1 + (1-a)p_2$.*

Proof. The expected return on portfolio p_3 is

$$\mu_3 = a\mu_1 + (1-a)\mu_2.$$

Setting $\mu_3 = \mu^*$, solve for a and verify that a is between 0 and 1:

$$\mu^* = a\mu_1 + (1-a)\mu_2,$$

$$a = \frac{\mu^* - \mu_2}{\mu_1 - \mu_2},$$

which is in [0,1] for any $\mu^* \in [\mu_1, \mu_2]$. This completes the proof. ■

From Lemma 9.1, we know that any portfolio of two efficient portfolios is itself efficient. From Lemma 9.2, we know that we can construct an efficient portfolio with any mean return between the two mean returns of the initial efficient portfolios.

Let p_{min} be the efficient portfolio with the least expected return of all efficient portfolios. Let p_{max} be the efficient portfolio with the highest expected return of all efficient portfolios. By assumption, any efficient portfolio has mean return between μ_{min} and μ_{max}. By Lemma 9.2, given any μ^* we can construct a portfolio from p_{min}, p_{max} with expected return μ^*. By Lemma 9.1, we know that this new portfolio is an efficient portfolio. It follows that we can generate the entire set of efficient portfolios from the two portfolios p_{min}, p_{max}. ■

Theorem 9.2 *The Second Mutual Fund Theorem. Consider a market with n risky assets available for investment. There also exists a risk free asset with constant return r_0. There exists an efficient risky portfolios (denoted p) such that any efficient portfolio p^* can be written as a portfolio of r_0 and p.*

Proof. The Second Mutual Fund Theorem can be thought of as a corollary of the First Mutual Fund Theorem. In this case, the risk free asset is the least return efficient portfolio (referred to as p_{min} in the proof of the First Mutual Fund Theorem). ■

Problems for Chapter 9

Exercise 9.1 *State and prove the First Mutual Fund Theorem.*

Exercise 9.2 *Do the First and Second Mutual Fund Theorem's help us explain trends in financial asset allocation?*

Exercise 9.3 *The First Mutual Fund Theorem appears to suggest that all investors should hold just two portfolios, with weights only dependent on their risk tolerance. If this is the case, why do some investors engage in stock-picking?*

Exercise 9.4 *Find the returns for two stocks online (for example, from* `finance.yahoo.com`*). Find the mean returns, variances and covariances for these two assets. Construct a two-asset portfolio frontier from this data, as in Figure 9.1.*

Exercise 9.5 *Is the following statement true or false? Explain your answer.*

In markets with a risk free asset, all mean-variance efficient portfolios have no unsystematic risk.

(Hint: What is the role of the risk free asset in your explanation?)

Coding exercises for Chapter 9

Coding Exercise 9.1 *The purpose of this exercise is to visualise the practical limits of diversification. First, you'll calculate covariances between stock returns for a selection of NYSE stocks. Then you'll use Propositions 9.1 and 9.1 to calculate the variances of equal weighted portfolios of uncorrelated securities and securities correlated as in the data.*

```
#-------------------------------------------------------------------------------
# Preamble
#-------------------------------------------------------------------------------

# Load packages

using UnicodePlots
using DataFrames

#-------------------------------------------------------------------------------
# Download and manipulate data
#
# We'll just use this data to get an average covariance between stocks.
# We can use this to approximate the gains from diversification across stocks.
#-------------------------------------------------------------------------------

println("Loading dataset.")
SP = DataFrame()
SP = readtable("StockPrices.csv")

# This data was originally downloaded from finance.google.com, although I've
# removed the script used for the download to save time.

# The date formatting is a bit unusual. We need to
# (a) Tell Julia how to read the dates by specifying a date format.
# (b) Change all the two-digit years into four-digit years.
# (c) Rename the date column.
# (d) Drop values older than 2 years, just to reduce the size of the dataset
#     and speed things up.

println("Formatting dates.")
# (a)
df          = Dates.DateFormat("dd-u-yy")
SP[:_Date] = Date(SP[:_Date],df)

# (b)
for i in 1:size(SP)[1]
  if Dates.Year(SP[:_Date][i]) <= (Dates.Year(Dates.today()) - Dates.Year(2000))
    SP[:_Date][i] = SP[:_Date][i] + Dates.Year(2000)
  else
    SP[:_Date][i] = SP[:_Date][i] + Dates.Year(1900)
  end
end

# (c)
rename!(SP,:_Date,:Date)

# (d)
println("Deleting old values")
SP = SP[SP[:Date] .> Dates.today() - Dates.Year(2),:]
println(size(SP)[1]," entries remaining.")

# Lets just keep twenty thousand entries and delete the rest
println("Keep first twenty thousand entries")
SP = SP[1:20000,:]

# We don't need the Volume, Open, High, and Low columns
delete!(SP,[:Open ; :High ; :Low ; :Volume])

# Generate monthly returns
# Note that this will take some time, there are many entries.
println("Calculating monthly returns")
SP[:Return] = 0.0
lastprint = 0;
```

```
for i in 1:size(SP)[1]
  try
    SP[:Return][i] = (
          log(SP[:Close][i])
          - log(SP[((SP[:Symbol] .== SP[:Symbol][i]) &
                    (SP[:Date] .== SP[:Date][i] - Dates.Month(1))),:][:Close][1])
                    )
  catch
    SP[:Return][i] = NA
  end
  try
  if SP[:Symbol][i+1] != SP[:Symbol][i] && i > lastprint + size(SP)[1]/20
    println("Current ticker: ",SP[:Symbol][i],
            " | ",round(100*i/size(SP)[1],0),"% completed.")
    lastprint = i;
  end
  end
end

# Delete NA values
println("Deleting NA values")
SP = SP[~isna(SP[:Return]),:]
println(size(SP)[1]," entries remaining.")

# Balance the panel by dropping symbols without a full complement of observations
for i in unique(SP[:Symbol])
  if size(SP[SP[:Symbol] .== i,:])[1] < size(unique(SP[:Date]))[1]
    SP = SP[SP[:Symbol] .!= i,:]
  end
end

#-------------------------------------------------------------------------------
# Calculate variances and covariances
#-------------------------------------------------------------------------------

# Calculate excess returns, x - mean(x) for each asset.
SP[:ER] = 0.0
for i in 1:size(SP[:Symbol])[1]
  SP[:ER][i] = (SP[:Return][i]-mean(SP[SP[:Symbol].==SP[:Symbol][i],:][:Return]))
end

# Initialise variance-covariance matrix
Mvar = zeros(size(unique(SP[:Symbol]))[1],
             size(unique(SP[:Symbol]))[1])

# Calculate covariance terms
for i in 1:size(unique(SP[:Symbol]))[1]
  for j in 1:size(unique(SP[:Symbol]))[1]
    if Mvar[j,i] != 0
      Mvar[i,j] = Mvar[j,i] # Exploit symmetry of the var-covar matrix
    else
      Mvar[i,j] = (sum(
                        (SP[SP[:Symbol].==unique(SP[:Symbol])[i],:][:ER]).*
                        (SP[SP[:Symbol].==unique(SP[:Symbol])[j],:][:ER]))/
                   size(unique(SP[:Date]))[1])
    end
  end
end

# Find the average covariance of pairs with i != j:
mucov = 0.5*sum(Mvar - diag(Mvar).*eye(Mvar))/(size(Mvar)[1]^2-size(Mvar)[1]);

# Find the average variance of individual stock returns
muvar = mean(diag(Mvar));

#-------------------------------------------------------------------------------
# Portfolio diversification and variances
#-------------------------------------------------------------------------------

# Set max number of assets
max_n = 20;

# (a) Uncorrelated assets (Proposition 7.1)

varpa = [muvar/n for n in 1:max_n];
```

```
# (b) Correlated assets (Prop 7.2)

varpb = [muvar/n + mucov*(n-1)/n for n in 1:max_n];

# (c) Limit as n -> infinity

varpc = mucov;

# Plot results

plotdiv = lineplot(collect(linspace(1,max_n,max_n)),varpa,
                   title="The limits of diversification");
  lineplot!(plotdiv,collect(linspace(1,max_n,max_n)),varpb);
  lineplot!(plotdiv,collect(linspace(1,max_n,max_n)),varpc*ones(max_n))
  ylabel!(plotdiv,"Portfolio variance")
  xlabel!(plotdiv,"number of securities")
```

10
The Capital Asset Pricing Model

Some investments do have higher expected returns than others.
Which ones?
Well, by and large they're the ones that will do the worst in bad times.
William Sharpe

Adding a risk-free asset

In the previous chapter we constructed the Markowitz bullet for a market with many assets. At the end of that lecture, we proved The Second Mutual Fund Theorem (Theorem 9.2).

The theorem stated that in a market with many risky assets and one risk free asset, we could construct any mean-variance efficient portfolio as a combination of the risk free asset and a single risky mutual fund.

So, how does this work graphically? First we need to introduce a risk free asset into our market. Figure 10.1 shows how we can construct portfolios from our risky assets and the risk free rate.

Consider a portfolio constructed from the risk free asset x_0 and some risky asset (or risky portfolio) x_j:

$$p = ax_0 + (1-a)x_j \tag{10.1}$$

The mean return on portfolio p is

$$\mu_p = ar_0 + (1-a)\mu_j. \tag{10.2}$$

The standard deviation of returns on portfolio p is

$$\sigma_p = \sqrt{a^2\sigma_0^2 + (1-a)^2\sigma_j^2} = (1-a)\sigma_j. \tag{10.3}$$

The relationship between μ_p and a is linear. As is the relationship between σ_p and a. It follows that the portfolios that we can construct trace a straight line in the (σ, μ) space between $(r_0, 0)$ and (μ_j, σ_j).

A sample of these portfolios combining risky assets with the risk free asset are presented as dashed lines in Figure 10.1.

Mean-variance efficient portfolios are in blue.
Dashed schedules indicate portfolios with both risk free and risky assets.

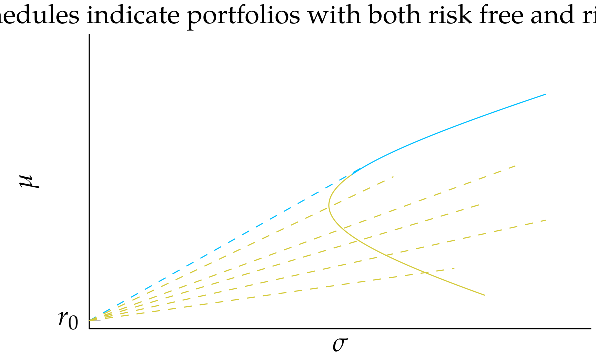


Figure 10.1: Adding a risk free asset.

Adding leverage

The portfolios constructed in Figure 10.1 contain positive proportions of the risk free asset, combined with risky assets and portfolios. If we can borrow at the risk free rate, then we can construct portfolios with mean returns exceeding those of the risky portfolio.

In mathematical terms, we can start from Equations 10.1,10.2 and 10.3. Using leverage means borrowing to fund investments in the risky portfolio. This corresponds to a negative value of portfolio weight a.

Graphically, Figure **??** shows how we can use leverage to extend the set of feasible portfolios. Figure **??** shows the set of portfolios that can be constructed from combinations of the risk free asset, and a particular portfolio of risky assets on the efficient frontier, referred to as the *tangency portfolio*. The tangency portfolio is marked by z in Figure **??**. The blue schedule traces these combinations, with the solid section tracing the combinations including a positive share of the risk free asset, the dashed section tracing the combinations that require leverage.

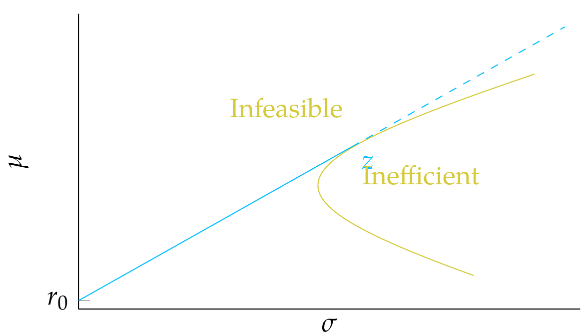


Figure 10.2: Efficient portfolios (blue) with leverage (dashed)

Should we add leverage?

Is it realistic to believe that individual investors can borrow unlimited amounts at the risk free rate?

The short answer is no. Certainly, if you or I were to try to borrow large amounts from the bank to speculate in stocks, we would be unlikely to be able to borrow at the risk free rate.

But, in order to determine whether or not this assumption really matters for our analysis, we need to consider the following two questions. First, how likely is it that investors would wish to use leverage? Second, can the payoffs of a levered portfolio be constructed without borrowing?

We'll take the first question first. Certainly, private sector investors hold positive amounts of risk free assets on average, with the government sector (typically) being a net supplier of risk free assets in the form of government bonds. It must then be the case that the average investor holds a positive proportion of their wealth in safe assets---in other words, the average investor does not wish to borrow, even at the risk free rate. This doesn't rule out the possibility that there are some investors who wish to use leverage, but it does suggest that most investors are happy not to use leverage.

Now, moving on to the second question. Consider an investor who cannot borrow, but would wish to hold a levered portfolio. Without access to borrowing at the risk free rate, this investor could still hold derivatives. Futures, forwards and options all have the property that they offer the investor exposure to movements in the underlying asset with little initial investment. In other words, these products provide a form of leverage, without the need for borrowing. Using derivatives, we can construct portfolios that replicate the payoffs of levered portfolios. It follows that it makes sense in our environment to allow for leverage.

To summarise, it is clear that most investors are unable to borrow unlimited amounts at the risk free rate. However, most investors would not wish to do so, and those who would wish to do so can replicate their desired portfolios by using derivatives. Therefore, the assumption of unlimited borrowing and lending at the risk free rate is not unreasonable for our purposes.

> I would never be 100 percent in stocks or 100 percent in bonds or cash.
> Harry Markowitz

Sharpe ratio

Definition 10.1 *The* Sharpe ratio *of asset i is defined as*

$$s_i = \frac{\mu_i - \mu_0}{\sigma_i}$$

Proposition 10.1 *Consider a market with many risky assets and unlimited borrowing and lending at the risk-free rate r_0. Let p_i and p_j be two mean-variance efficient portfolios. The Sharpe ratios s_i and s_j are equal:*

$$s_i = s_j.$$

Alternatively,

$$\frac{\mu_i - \mu_0}{\sigma_i} = \frac{\mu_j - \mu_0}{\sigma_j}.$$

Proof is left to student.

Proposition 10.2 *Consider a market with many risky assets and unlimited borrowing and lending at the risk-free rate r_0. Let p_i be a mean-variance* *efficient portfolio. Let p_j be a mean-variance inefficient portfolio. The Sharpe ratio of p_i is greater than the Sharpe ratio of p_j,*

$$s_i > s_j.$$

Alternatively,

$$\frac{\mu_i - \mu_0}{\sigma_i} > \frac{\mu_j - \mu_0}{\sigma_j}.$$

Proof is left to student.

The Capital Market Line

The Assuming unlimited borrowing and lending at the risk free rate, the schedule of efficient portfolios is linear and upward sloping in the (σ, μ) space with vertical intercept at the risk free rate r_0. This schedule is referred to as the *Capital Market Line*. The point on the Capital Market Line where there is neither borrowing or lending at the risk free rate is referred to as the *Market Portfolio*. This is the point at which the portfolio weight on the portfolio of risky assets is 1. By the Second Mutual Fund Theorem, every point on the Capital Market Line is a combination of the Market Portfolio m and the risk free asset.

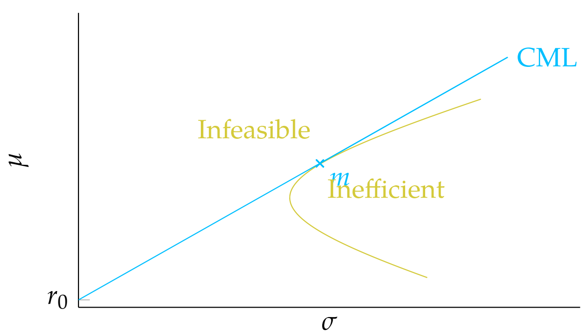


Figure 10.3: The Capital Market Line

All assets on the Capital Market Line have the same Sharpe ratio. Any asset below the Capital Market Line has a Sharpe ratio lower than the Sharpe ratio of any asset on the Capital Market Line. The slope of the Capital Market Line is equal to the Sharpe ratio of all efficient assets.

The Capital Asset Pricing Model

Theorem 10.1 *The Capital Asset Pricing Model formula. Let μ_j denote the expected return of asset j, and μ_m denote the expected return of the market portfolio (the tangency portfolio). The risk free rate is denoted by r_0, and* *investors are able to borrow and lend unlimited amounts at the risk free rate. The expected return on asset j can be described as follows:*

$$\mu_j = r_0 + \beta_j(\mu_m - r_0), \tag{10.4}$$

where $\beta_j = \dfrac{cov(r_j, r_m)}{var(r_m)}$.

Lets think about what this is saying. The expected return on asset j depends *only* on the risk free rate, the expected return for the market portfolio, the covariance of asset j returns with market returns and the variance of market returns.

Remember, asset pricing is about beliefs, preferences and arbitrage. In the FVR, investors' preferences entered the model explicitly. In the CAPM, preferences are proxied by the market risk premium $\mu_m - r_0$, which captures how averse investors are to fluctuations in the overall stock market. Beliefs still enter the model explicitly through β and μ_m, both expectations.

What does this list exclude? Lots of things! For example, the CAPM predicts that the idiosyncratic component of an asset's risk does not influence the expected returns of the asset. Only the systematic, or correlated component of the asset's risk matters.

Proof. First, we need to reconsider the Fundamental Valuation Relationship from Chapter 7. We can write down the Fundamental Valuation Relationship as follows

$$v'(w) = \mathbb{E}[u'(c)(1 + r_j)] \qquad \forall j$$

where $v'(w)$ is the marginal value of initial wealth, $u'(c)$ is the realised marginal utility of consumption, and $(1 + r_j)$ is the realised gross return to asset j. Consider the term in the expectation on the right hand side, $u'(c)(1 + r_j)$ is the marginal contribution of an additional unit of asset j to realised utility. The expectation $\mathbb{E}[u'(c)(1 + r_j)]$ is the expected increase in expected utility accruing from a marginal increase in holdings of asset j.

The key message of the Fundamental Valuation Relationship is that this marginal increase in expected utility accruing from additional holdings of assets is the same for all assets. If it were the case that

$$\mathbb{E}[u'(c)(1 + r_i)] < \mathbb{E}[u'(c)(1 + r_j)],$$

then we should sell some of asset i and buy some of asset j.

OK, what does this mean for us? Let z be an investor's optimal portfolio, and let x_j be a risky asset that has positive weight in portfolio z. The fundamental valuation relationship tells us that at the margin, shifting wealth from portfolio z to asset x_j should have no effect on expected utility ($\mathbb{E}[u'(c)(1 + r_z)] = \mathbb{E}[u'(c)(1 + r_j)]$). At the margin, small changes in the holdings of both assets have the same effect on expected utility.

Graphically, this means that starting from optimal portfolio z, if we construct portfolios from z and x_j, the schedule of portfolios should be tangent to the investor's indifference curves when the weight on portfolio z is 1. Figure **??** plots this scenario. The green schedule I

is the investor's indifference curve. The blue schedule is the Capital Markets Line, tracing the efficient portfolios. The investor's optimal portfolio is marked by z. The red curve illustrates the portfolios that can be constructed from z and x_j. At z, this portfolio is tangent to the Capital Markets Line and the investor's indifference curve if and only if the asset x_j is in the optimal portfolio z.

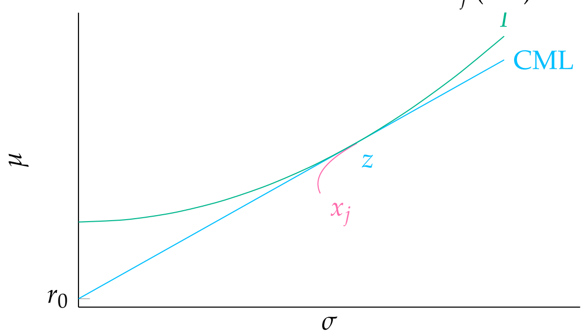


Figure 10.4: Efficient portfolio frontier (blue)

Let p be a portfolio constructed from the investor's optimal portfolio z, and the risky asset x_j,

$$p = ax_j + (1-a)z$$

The expected return and standard deviation for portfolio p are

$$\mu_p = a\mu_j + (1-a)\mu_z$$

$$\sigma_p = \sqrt{a^2\sigma_j^2 + (1-a)^2\sigma_z^2 + 2a(1-a)\sigma_{jz}}.$$

We know that when a is close to zero, the slope traced out by the portfolios p in the volatility, mean returns space (σ, μ) is equal to the slope of the Sharpe Ratio of portfolio z. That is,

$$\lim_{a \to 0} \frac{d\mu_p}{d\sigma_p} = s_z. \tag{10.5}$$

Lets consider the left hand side of Equation **??**. Using the chain rule, we know that

$$\frac{d\mu_p}{d\sigma_p} = \frac{d\mu_p}{da} \bigg/ \frac{d\sigma_p}{da}.$$

We can solve this, first taking the derivatives with respect to a.

$$\frac{d\mu_p}{da} = \mu_j - \mu_z$$

$$\frac{d\sigma_p}{da} = \frac{2a\sigma_j^2 - 2(1-a)\sigma_z^2 + 2(1-2a)\sigma_{jz}}{2\sqrt{a^2\sigma_j^2 + (1-a)^2\sigma_z^2 + 2a(1-a)\sigma_{jz}}}$$
$$= \frac{a\sigma_j^2 - (1-a)\sigma_z^2 + (1-2a)\sigma_{jz}}{\sqrt{a^2\sigma_j^2 + (1-a)^2\sigma_z^2 + 2a(1-a)\sigma_{jz}}}$$

This looks ugly, but remember that we are just interested in the limit as a approaches zero. Most of the terms in $\partial\sigma_p/\partial a$ are just going to cancel out.

$$\lim_{a\to 0} \frac{d\mu_p}{da} = \mu_j - \mu_z$$

$$\lim_{a\to 0} \frac{d\sigma_p}{da} = \frac{-\sigma_z^2 + \sigma_{jz}}{\sqrt{\sigma_z^2}} = \frac{\sigma_{jz} - \sigma_z^2}{\sigma_z}.$$

It follows that

$$\lim_{a\to 0} \frac{d\mu_p}{d\sigma_p} = \frac{\sigma_z}{\sigma_{jz} - \sigma_z^2}(\mu_j - \mu_z) \tag{10.6}$$

Now, lets consider the right hand side of Equation **??**. The Sharpe ratio s_z is equal to

$$s_z = \frac{\mu_z - r_0}{\sigma_z}. \tag{10.7}$$

Now we can substitute (**??**) and (**??**) into (**??**),

$$\frac{\sigma_z}{\sigma_{jz} - \sigma_z^2}(\mu_j - \mu_z) = \frac{\mu_z - r_0}{\sigma_z}.$$

Rearranging yields

$$\mu_j - \mu_z = (\mu_z - r_0)\left(\frac{\sigma_{jz} - \sigma_z^2}{\sigma_z^2}\right),$$

$$\mu_j - \mu_z = (\mu_z - r_0)\left(\frac{\sigma_{jz}}{\sigma_z^2} - 1\right),$$

$$\mu_j = r_0 + \frac{\sigma_{jz}}{\sigma_z^2}(\mu_z - r_0).$$

This relationship holds for any mean-variance efficient portfolio z. The market portfolio m is a mean-variance efficient portfolio, therefore

$$\mu_j = r_0 + \frac{\sigma_{jm}}{\sigma_m^2}(\mu_m - r_0),$$

$$\mu_j = r_0 + \beta_j(\mu_m - r_0).$$

We have shown that any risky asset in the market portfolio must satisfy the CAPM equation. Any asset that does not satisfy the CAPM must not be in the market portfolio, and consequently will not be held by any mean-variance optimising investor. ∎

The Security Market Line

Figure **??** plots the Security Market Line for a market with many risky assets. The Security Market Line (SML or Characteristic Line) is a graphical representation of the CAPM equation, $\mu_j = r_0 + \beta_j(\mu_m - r_0)$, plotted in the (β, μ_j) space. The vertical intercept is r_0, the expected return of a zero-β asset. The slope of the SML is the market risk premium, $(\mu_m - r_0)$. The market portfolio m has beta 1 and expected return μ_m, and sits on the Security Market Line.

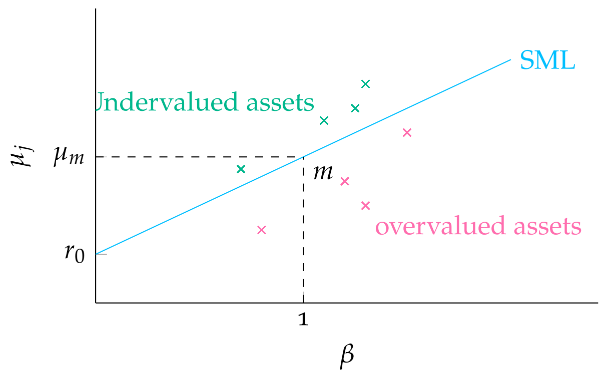


Figure 10.5: The Security Market Line

Assets sitting above the Security Market Line are undervalued with respect to the CAPM. Their expected return is high relative to their systematic risk (recall that β is a measure of the systematic component of an asset's risk). Assets sitting below the Security Market Line are overvalued with respect to the CAPM. Their expected return is low relative to their systematic risk. As we can see in Figure **??**, even assets with expected return less than the market return can be undervalued. Similarly, even assets with expected return exceeding the market return can be overvalued.[1]

[1] That is, there are green assets below μ_m, and red assets above μ_m.

Determination of the market risk premium

Consider the CAPM formula (**??**),

$$\mu_j = r_0 + \beta_j(\mu_m - r_0).$$

The term in brackets on the right hand side, $\mu_m - r_0$, is the *market risk premium*, the expected excess return of the market portfolio over the risk free rate.

In Lecture 7, we derived the Fundamental Valuation Relationship (Theorem 7.1). Corollary 7.1 showed us that the expected excess return of an asset over the risk free rate is determined by the covariance between the return of the asset and the marginal utility of the investor.

From Equation 7.7 we have

$$\mu_j = r_0 - \frac{\text{cov}(u'(c), r_j)}{\mathbb{E}[u'(c)]}.$$

This relationship must also hold for the market portfolio:

$$\mu_m - r_0 = -\frac{\text{cov}(u'(c), r_m)}{\mathbb{E}[u'(c)]}.$$

The market risk premium is driven by the covariance of market returns with consumption marginal utility. Typically, consumption and equity returns will move together, rising in booms and falling in downturns (we can see this positive correlation between consumption and returns to equity in Figure 7.1). This motivates a negative covariance between consumption marginal utility and equity returns. It is this negative covariance between consumption marginal utility and equity returns that determine the market risk premium.

Next term, you'll study the Consumption-CAPM, a version of the CAPM that further develops this link between consumption volatility and the market risk premium.

How can we test the CAPM?

Consider the following regression model:

$$(r_{jt} - r_{0t}) = \alpha_j + \beta_j(r_{mt} - r_{0t}) + \gamma_j(r_{mt} - r_{0t})^2 + \Delta_j \Omega'_{t-1} + \varepsilon_{jt}, \quad (10.8)$$

where Ω_{t-1} is a vector of information known in period $t-1$. This could include past returns for asset j, it could also include accounting information for asset j. The term Δ_j represents a vector of coefficients δ_{kj} corresponding to individual elements ω_{kt-1} of the information set Ω_{t-1}.

The CAPM predicts that the coefficient $\hat{\beta}_j$ should be significant and non-zero. Significant, non-zero values for the coefficients $\hat{\alpha}_j, \hat{\gamma}_j, \hat{\delta}_j$ would be evidence against the CAPM.

If the coefficient $\hat{\alpha}_j$ were found to be significantly different from zero, this would suggest that the intercept of the Capital Market Line is not the risk free rate. For example, ff $\hat{\alpha}_j$ were found to be positive, this would suggest that stocks are generally undervalued when compared with the prediction of the CAPM. If $\hat{\gamma}_j$ were found to be significantly different from zero, this would suggest that the relationship between expected returns and β is non-linear. If $\hat{\delta}_{kj}$ were found to be significantly different from zero, this would suggest that past information is predictive of future relative performance of stocks, adjusted for risk. Any of these would be evidence against the CAPM.

So, what happens when we do test the CAPM? Bailey (2005, Ch. 9) has an excellent review. There are other sources provided on Moodle.

Application of the CAPM: Network regulation

From your reading of Bailey (2005, Ch. 9), you will be well aware that the CAPM does not perform well in empirical tests. The performance of the CAPM can be improved by extending the baseline model to account for taxes, time varying β coefficients and time varying market risk premia, but even with these extensions the CAPM has less predictive power than atheoretical statistical models. Nevertheless, the CAPM still has some useful applications, one of which is in network regulation.

Consider a natural monopoly network industry. This could be a pipeline network carrying oil or gas. It could be an electricity lines network, it could be a broadband network. Each of these networks are natural monopolies, it is inefficient to have many competing networks. These networks share a high initial cost of construction, with typically a low short run marginal cost of providing additional service on the network to consumers.

Network industries are typically subject to price regulation. From ECON 1 we know that without regulation, the operating monopolies would have an incentive to set prices above long run marginal costs, resulting in inefficiently low output. Regulators aim to set prices near long run marginal cost, where long run marginal cost includes both the short run costs of maintaining the network and a market return to the initial investment. This pricing would eliminate monopoly rents while still providing the incentive to invest in new network capacity when it is efficient to do so. If the regulated price is too low, there will be no incentive for the network operator to invest in expanding the network, even when the current network has reached capacity. If the regulated price is too high, the firm will capture monopoly rents and the market output will be below the efficient level.

The short run component of marginal cost is reasonably straightforward for regulators to measure. But, the long run component of long run marginal costs, the fair return to the fixed capital investment, is more difficult to measure. Regulators need to use an asset pricing model to determine the fair return to fixed capital investment. The CAPM is a popular model for this purpose. While the CAPM has less predictive power than some atheoretical models, the CAPM is a transparent model with few inputs, and these inputs (β, the market risk premium) are difficult to manipulate.

Problems for Chapter 10

Exercise 10.1 *Prove Propositions* **??** *and* **??**.

Exercise 10.2 *For each of the following examples, state whether and explain why the information provided results in a contradiction. Use diagrams.*

Assets x_i and x_j are both mean-variance efficient assets. r_0 is the risk free interest rate:

a. $r_0 = 2\%, \mu_i = 5\%, \sigma_i = 10\%, \mu_j = 6\%, \sigma_j = 15\%$.

b. $r_0 = 2\%, \mu_i = 5\%, \sigma_i = 10\%, \mu_j = 6\%, \sigma_j = 9\%$.

c. $r_0 = 2\%, \mu_i = 5\%, \sigma_i = 10\%, \mu_j = 8\%, \sigma_j = 20\%$.

d. $\mu_i = 5\%, \sigma_i = 10\%, \mu_j = 6\%, \sigma_j = 15\%, \mu_k = 8\%, \sigma_k = 20\%$.

Exercise 10.3 *State and prove the Capital Asset Pricing Model Theorem.*

Exercise 10.4 *What are the main predictions of the Capital Asset Pricing Model? Are these predictions supported empirically?*

Exercise 10.5 *Consider the Capital Asset Pricing Model equation:*

$$\mu_j = r_0 + \beta_j(\mu_m - r_0),$$

where μ_j is the expected return on asset j, r_0 is the risk free rate, the coefficient $\beta_j = \frac{cov(r_j, r_m)}{var(r_m)}$ and μ_m is the expected return on the market portfolio.

a. Explain why μ_j is increasing in β_j.

b. Explain why idiosyncratic risk does not directly affect expected returns predicted by the CAPM.

c. What factors determine the market risk premium, $\mu_m - r_0$?

Exercise 10.6 *The Capital Asset Pricing Model is widely used to determine the market return to fixed assets in network monopoly regulation. What are the strengths and weaknesses of the Capital Asset Pricing Model for this application?*

Exercise 10.7 *This question is concerned with the assumption of unlimited borrowing and lending at the risk free rate. This assumption is relied upon to prove the Capital Asset Pricing Model.*

a. Sketch the efficient frontier for an investor with unlimited borrowing and lending at the risk free rate.

b. Sketch the efficient frontier for an investor with a low interest rate on savings and a high interest rate on borrowing.

c. *How realistic is the assumption of unlimited borrowing and lending at the risk free rate?*

d. *Explain, with a specific example, how an investor can use derivatives to replicate levered portfolios, without the explicit need for borrowing.*

Coding exercise for Chapter 10

Coding Exercise 10.1 *In this exercise, you will calculate realised beta coefficients for a set of securities traded on the New York Stock Exchange. Then, you will plot the realised mean returns for each security against their realised beta coefficients.*

How does this plot relate to the Security Market Line? Do your results support the CAPM model?

```
#-------------------------------------------------------------------------------
# Preamble
#-------------------------------------------------------------------------------

# Load packages

using Gadfly
using DataFrames

#-------------------------------------------------------------------------------
# Download and manipulate data
#
# We'll just use this data to get an average covariance between stocks.
# We can use this to approximate the gains from diversification across stocks.
#-------------------------------------------------------------------------------

println("Loading dataset.")
SP = DataFrame()
SP = readtable("StockPrices.csv")

# This data was originally downloaded from finance.google.com, although I've
# removed the script used for the download to save time.

# The date formatting is a bit unusual. We need to
# (a) Tell Julia how to read the dates by specifying a date format.
# (b) Change all the two-digit years into four-digit years.
# (c) Rename the date column.

println("Formatting dates.")
# (a)
df          = Dates.DateFormat("dd-u-yy")
SP[:_Date] = Date(SP[:_Date],df)

# (b)
for i in 1:size(SP)[1]
  if Dates.Year(SP[:_Date][i]) <= (Dates.Year(Dates.today()) - Dates.Year(2000))
    SP[:_Date][i] = SP[:_Date][i] + Dates.Year(2000)
  else
    SP[:_Date][i] = SP[:_Date][i] + Dates.Year(1900)
  end
end

# (c)
rename!(SP,:_Date,:Date)

# We don't need the Volume, Open, High, and Low columns
delete!(SP,[:Open ; :High ; :Low ; :Volume])

# Read in total market data

RussellTR        = readtable("RussellTR.csv")
dfm              = Dates.DateFormat("yyyy-mm-dd")
RussellTR[:date] = Date(RussellTR[:date],dfm)

RussellTR      = RussellTR[RussellTR[:value].>0,:]     # Remove NaN values
rename!(RussellTR,[:date ; :value],[:Date ; :Value])  # Comparable col names

#-------------------------------------------------------------------------------
# The dataset is too large to work with effectively.
# We'll aggregate by end-of-month price.
#-------------------------------------------------------------------------------
```

```
# First we need to identify end-of-month dates.

SP[:EOM]    = 0
SP[:EOM][1] = 1
for i in 2:size(SP)[1]
  if (SP[:Symbol][i] == SP[:Symbol][i-1] &&
      Dates.Day(SP[:Date][i]) > Dates.Day(SP[:Date][i-1]))
      SP[:EOM][i] = 1
  end
end

# Now delete all dates that are not end-of-month

SP = SP[SP[:EOM] .== 1,:]
delete!(SP,[:EOM])

println("After aggregating months, there are")
println(size(SP)[1],
        " observations remaining for ",
        size(unique(SP[:Symbol]))[1],
        " securities.")

# Now do the same for the index

RussellTR[:EOM]    = 0
RussellTR[:EOM][size(RussellTR)[1]] = 1
for i in 1:size(RussellTR)[1]-1
  if (Dates.Day(RussellTR[:Date][size(RussellTR)[1] - i])
      > Dates.Day(RussellTR[:Date][size(RussellTR)[1] - (i-1)]))
      RussellTR[:EOM][size(RussellTR)[1] - i] = 1
  end
end
RussellTR = RussellTR[RussellTR[:EOM] .== 1,:]
delete!(RussellTR,[:EOM])

#-------------------------------------------------------------------------------
# Generate monthly returns series
#-------------------------------------------------------------------------------

SP[:RM] = 0.0                                  # Initialise return column
for i in 1:size(SP)[1]
  try
    if   SP[:Symbol][i] == SP[:Symbol][i+1]
         SP[:RM][i] = (SP[:Close][i] - SP[:Close][i+1])/SP[:Close][i+1]
    else SP[:RM][i] = NA
    end
  catch  SP[:RM][i] = NA
  end
end

# Drop NA values
SP = SP[.~isna.(SP[:RM]),:];

RussellTR[:RM] = 0.0                           # Initialise return column
for i in 1:size(RussellTR)[1]
  try
    RussellTR[:RM][i] = (RussellTR[:Value][i] - RussellTR[:Value][i-1])/RussellTR[:Value][i-1]
  catch  RussellTR[:RM][i] = NA
  end
end

# Drop NA values
RussellTR = RussellTR[.~isna.(RussellTR[:RM]),:];


# Drop observations over 10 years old, drop securities with less than
# 10 years of observations.
SP = SP[SP[:Date] .> Dates.today() - Dates.Year(10),:]
for i in unique(SP[:Symbol])
  if size(SP[SP[:Symbol] .== i,:])[1] < size(unique(SP[:Date]))[1]
    SP = SP[SP[:Symbol] .!= i,:]
  end
end

println("After dropping short lived securities, there are")
```

```
println(size(SP)[1],
         " observations remaining for ",
         size(unique(SP[:Symbol]))[1],
         " securities.")

RussellTR = RussellTR[((RussellTR[:Date] .<= maximum(unique(SP[:Date]))) .&
                       (RussellTR[:Date] .>= minimum(unique(SP[:Date])))),:]

#--------------------------------------------------------------------------------
# Calculate variances and covariances
#--------------------------------------------------------------------------------

# Calculate excess returns, x - mean(x) for each asset.
SP[:ER] = 0.0
for i in 1:size(SP[:Symbol])[1]
  SP[:ER][i] = (SP[:RM][i]-mean(SP[SP[:Symbol].==SP[:Symbol][i],:][:RM]))
end

RussellTR[:ER] = 0.0
for i in 1:size(RussellTR)[1]
  RussellTR[:ER][i] = (RussellTR[:RM][i]-mean(RussellTR[:RM]))
end

# Add market return to SP dataframe
SP[:RZ] = 0.0
for i in 1:size(SP)[1]
  try   SP[:RZ][i] = RussellTR[RussellTR[:Date] .== SP[:Date][i],:][:RM][1]
  catch SP[:RZ][i] = NA;
  end
end

# Delete NA values
SP = SP[.~isna.(SP[:RZ]),:];

# Calculate betas
beta = zeros(size(unique(SP[:Symbol]))[1]);
for i in 1:size(unique(SP[:Symbol]))[1]
  symbol = unique(SP[:Symbol])[i]
  bcov   = cov(SP[SP[:Symbol].==symbol,:][:RM],SP[SP[:Symbol].==symbol,:][:RZ]);
  bvar   = var(SP[SP[:Symbol].==symbol,:][:RZ]);
  beta[i]= bcov/bvar;
end



# Calculate expected returns
mur = zeros(size(unique(SP[:Symbol]))[1]);
for i in 1:size(unique(SP[:Symbol]))[1]
  symbol = unique(SP[:Symbol])[i]
  mur[i] = mean(SP[SP[:Symbol].==symbol,:][:RM]);
end

# fit linear model to betas / expected returns
x     = [ones(beta) beta];
theta =  (x'*x)\x'*mur;

a = Scale.color_discrete_hue();
#define_color("color2", a.f(3)[2]);

plotbetamur = plot(layer(x = beta,
                         y = mur,
                         Geom.point,
                         intercept=[theta[1]], slope=[theta[2]],
                         Geom.abline),
                   layer(x=beta,y=mur,intercept=[0], slope=[mean(RussellTR[:RM])],
                         Geom.abline(color=a.f(3)[2])),
                   Guide.title("Security market line: data vs model prediction"),
                   Guide.xlabel("security beta"),
                   Guide.ylabel("mean return"),
                   Coord.cartesian(xmin=0.0,xmax=3,ymin=-0.01,ymax=0.03),
                   Guide.manual_color_key("", ["Data","CAPM"], [a.f(3)[1],a.f(3)[2]]),
                          )
draw(PNG("plot_capm.png", 4inch, 3inch), plotbetamur)
```

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