Optimal Macro-Financial Policies in a New Keynesian Model with Privately Optimal Risk Taking Online Appendix *

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A Financial Friction Derivations

We start with the model of Duncan and Nolan (2019), which predicts the following relationship between leverage, the factor wedge and uncertainty in the case of two income states and low audit costs:

$$L = \frac{(\bar{\pi} + \underline{\pi}\,\eta)T}{(\bar{\pi}\,\Xi - T)(\underline{\pi}\,\eta\Xi + T)} \tag{A.1}$$

where L is leverage, and Ξ is uncertainty measured as the difference between high and low income states, such that l, ξ denote log-linearised fluctuations in L, Ξ, and τ denotes linearised fluctuations in T.[1]

The allocations resulting from Equation A.1 are well approximated around the steady state by the limit as $\eta \to 0^+$.
Let $\eta \to 0$:

$$L = \frac{\bar{\pi}}{\bar{\pi}\,\Xi - T} \tag{A.2}$$

Taking log-quadratic approximations around the steady state yields

$$\begin{aligned} l = & \left(\frac{1}{\bar{\pi}\,\Xi - T}\right)\tau - \frac{\bar{\pi}\,\Xi}{\bar{\pi}\,\Xi - T}\xi \\ & + \frac{1}{2}\frac{1}{(\bar{\pi}\,\Xi - T)^2}\tau^2 + \frac{1}{2}\frac{\bar{\pi}\,\Xi T}{(\bar{\pi}\,\Xi - T)^2}\xi^2 - \frac{\bar{\pi}\,\Xi}{(\bar{\pi}\,\Xi - T)^2}\xi\tau + \mathcal{O}(z^3) \end{aligned}$$

$$\tau = (\bar{\pi}\,\Xi - T)l + \bar{\pi}\,\Xi\xi - \frac{1}{2}(\bar{\pi}\,\Xi - T)l^2 + \frac{1}{2}\bar{\pi}\,\Xi\xi^2 + \mathcal{O}(z^3)$$

which we express as follows:

$$\tau = \theta_l\left(l - \frac{1}{2}l^2\right) + \theta_\xi\left(\xi + \frac{1}{2}\xi^2\right) + \mathcal{O}(z^3) \tag{A.3}$$

where $\theta_l = \bar{\pi}\,\Xi - T$, and $\theta_\xi = \bar{\pi}\,\Xi$.

[1] Note that $\Xi \propto \sigma(\theta)$. As a result, log-linearised fluctuations in Ξ can be interpreted as equivalent to log-linear fluctuations in the volatility of idiosyncratic productivity shocks.

A.1 Equity risk premium

The equity risk premium is denoted as follows:

$$R = 1 + LT$$

and permits the following second order approximation:

$$\rho_t = \frac{L}{1+LT}\left(T\left(l_t + \frac{1}{2}l_t^2\right) + \tau_t + l_t\tau_t\right) + \mathcal{O}(z^3). \tag{A.4}$$

Using (A.3), we can write

$$\begin{aligned}\rho_t &= \frac{L}{1+LT}\left(\bar{\pi}\,\Xi l_t + \frac{1}{2}Tl_t^2 - \frac{1}{2}(\bar{\pi}\,\Xi - T)l^2 + \bar{\pi}\,\Xi\xi + \frac{1}{2}\bar{\pi}\,\Xi\xi^2 + \big((\bar{\pi}\,\Xi - T)l^2 + \bar{\pi}\,\Xi l\xi\big)\right) + \mathcal{O}(z^3) \\ &= \frac{\bar{\pi}\ + LT}{1+LT}\left(l_t + \frac{1}{2}l^2 + \xi + \frac{1}{2}\xi^2 + l\xi\right) + \mathcal{O}(z^3)\end{aligned}$$

which we denote

$$\rho_t = \psi\left(l_t + \frac{1}{2}l^2 + \xi + \frac{1}{2}\xi^2 + l\xi\right) + \mathcal{O}(z^3) \tag{A.5}$$

where $\psi = \frac{\bar{\pi}\ + LT}{1+LT}$.

A.2 The importance of audit signal errors η for quantitative analysis

Four our quantitative analysis, we use the full expression (A.1), with strictly positive η, which we allow to be estimated. The following broad relationships still hold, for the full nonlinear model,

$$\theta_l = \theta_\xi - T, \qquad \psi = \frac{L\theta_\xi}{1+LT}.$$

This is convenient, and makes the above limiting approximations useful for analysis of model dynamics. For policy analysis, we require strictly positive η. The expected welfare costs of audit errors for entrepreneurs do not vanish as $\eta \to 0^+$.

B Derivations of key model equations

B.1 The model in full

The IS curve

$$c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}]) \tag{B.1}$$

The Phillips curve

$$\pi_t = \beta\mathbb{E}_t[\pi_{t+1}] + \lambda \text{pp}_t \tag{B.2}$$

Risk Sharing

$$\sigma c_t - c_t^e = \sigma c_{t-1} - c_{t-1}^e - \rho_t - \delta_t \tag{B.3}$$

Aggregate Demand

$$x_t = \frac{\bar{c}}{\bar{x}}c_t + \frac{\bar{c}^e}{\bar{x}}c_t^e \tag{B.4}$$

Risk premia

$$\rho_t = \frac{LT}{1+LT}l_t + \frac{L}{1+LT}\tau_t \tag{B.5}$$

Leverage

$$x_t = c_t^e - \rho_t + l_t \tag{B.6}$$

Producer prices (marginal costs)

$$\text{pp}_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)x_t - \frac{1+\varphi}{1-\alpha}a_t + \sigma\omega(1-\psi)l_t - \sigma\omega\psi\xi_t + \tau_t \tag{B.7}$$

Labour wedge

$$\tau_t = \theta_l l_t + \theta_\xi \xi_t \tag{B.8}$$

Interest rate policy

$$i_t = \phi_\pi \pi_t + \phi_x x_t + \phi_l l_t \tag{B.9}$$

Prudential policy

$$\mathbb{E}_t[\delta_{t+1}] = 0 \tag{B.10}$$

B.2 Retailers

The final consumption goods consumed by households and entrepreneurs represent baskets over differentiated consumption goods. Aggregate consumption is given by

$$C = c + c^e$$

where

$$C = \left[\int_0^1 C(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

with $C(i)$ representing the quantity of good i consumed by all agents in the period.[2] The resulting demand schedule for individual consumption goods is as follows:

$$C(i) = \left(\frac{P(i)}{P}\right)^{-\varepsilon} C.$$

Differentiated final consumption goods $C(i)$ are produced by a continuum of retailers from the undifferentiated output goods sold by entrepreneurs. Retailers do not require labour or capital, and are owned by the representative worker household.

Following Calvo (1983), a retailer in period t can reset their price in the current (or any future period) with probability θ. The retailer solves the following programme:

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[m_{t,t+k} (P_t^* - \text{PP}_{t+k|t}) Y_{t+k|t} \right],$$

where m denotes the worker household's stochastic discount factor. Ultimately, retailer optimisation leads to the following log-linearised Phillips curve,

$$\pi_t = \beta \mathbb{E}_t[\pi_{t+1}] + \lambda \text{pp}_t$$

where $\lambda = \frac{(1-\theta)(1-\beta\theta)}{\theta} \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}$, π_t is the current period inflation rate and pp_t is the log deviation of producer prices from their steady state level.

[2]An implicit assumption here is that while entrepreneurs and households have different preferences over their overall level of consumption, they share common elasticities of substitution between constituent differentiated consumption goods.

B.3 Derivation of producer prices

Entrepreneurial output is homogeneous and priced competitively, with producer prices equated to marginal costs including the marginal cost of risk bearing:

$$\begin{aligned}
\mathrm{pp}_t &= w_t - (mpn_t - \tau_t) \\
&= (\sigma c_t + \varphi n_t) - (x_t - n_t - \tau_t) \\
&= \sigma c_t + \frac{1+\varphi}{1-\alpha}(x_t - a_t) - x_t + \tau_t
\end{aligned}$$

Substituting out consumption for terms of output, leverage and the equity risk premium yields

$$\mathrm{pp}_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) x_t - \frac{1+\varphi}{1-\alpha} a_t + (\theta_l + \sigma\omega(1-\psi)) l_t + (\theta_\xi - \sigma\omega\psi)\xi_t. \quad \text{(B.11)}$$

In addition to the standard New Keynesian marginal cost terms in output and technology, leverage and uncertainty increase the marginal cost of risk bearing $(\theta_l l_t, \theta_\xi \xi_t)$, and also affect the distribution of consumption, generating a wealth effect on marginal costs that is increasing in leverage and decreasing in uncertainty $(\sigma\omega(1-\psi)l_t, -\sigma\omega\psi\xi_t)$.

B.4 Derivation of the Leverage curve (Equation 1.3)

From equations B.4 and B.6 we obtain

$$c_t = x_t - \omega(\rho_t - l_t)$$

where $\omega := \frac{\bar{c}^e}{\bar{c}}$. Substituting this expression and (B.6) into the aggregate risk sharing relationship (B.3), we obtain

$$\begin{aligned}
&\sigma(x_t - \omega(\rho_t - l_t)) - (x_t + \rho_t - l_t) \\
&\qquad = \sigma(x_{t-1} - \omega(\rho_{t-1} - l_{t-1})) - (x_{t-1} + \rho_{t-1} - l_{t-1}) - \rho_t.
\end{aligned}$$

Collecting like terms,

$$(\sigma-1)x_t - \sigma\omega\rho_t + (\sigma\omega+1)l_t$$
$$= (\sigma-1)x_{t-1} - (\sigma\omega+1)(\rho_{t-1} - l_{t-1}).$$

$$(\sigma-1)x_t - \sigma\omega\rho_t + (\sigma\omega+1)l_t$$
$$= (\sigma-1)x_{t-1} - (\sigma\omega+1)(\rho_{t-1} - l_{t-1}).$$

Use (B.5,B.8,A.4) to eliminate ρ

$$\rho_t = \psi(l_t + \xi_t) \tag{B.12}$$

$$(\sigma-1)x_t - \sigma\omega(\psi l_t + \psi\xi_t) + (\sigma\omega+1)l_t$$
$$= (\sigma-1)x_{t-1} - (\sigma\omega+1)(\psi l_{t-1} + \psi\xi_{t-1} - l_{t-1}).$$

Simplifying yields

$$\zeta l_t = (\zeta-\psi)l_{t-1} + \sigma\omega\psi\xi_t - (\sigma\omega+1)\psi\xi_{t-1} - (\sigma-1)(x_t - x_{t-1}). \tag{B.13}$$

where $\zeta := (1+\sigma\omega(1-\psi))$.

B.5 Derivation of the IS curve (Equation 1.1)

Start by substituting the Aggregate Demand B.4, use Equations B.5,B.6,B.8 to eliminate entrepreneurial consumption c^e and express aggregate demand x in terms of household consumption c, leverage l and risk ξ only:

$$x_t = \frac{\bar{c}}{\bar{x}}c_t + \frac{\bar{c}^e}{\bar{x}}\left(x_t + \psi(l_t + \xi_t) - l_t\right).$$

Simplifying yields

$$c_t = x_t + \omega(1-\psi)l_t - \omega\psi\xi_t \tag{B.14}$$

Use this expression to find expected future consumption $\mathbb{E}[c_{t+1}]$:

$$\mathbb{E}[c_{t+1}] = \mathbb{E}[x_{t+1}] + \omega(1-\psi)\mathbb{E}[l_{t+1}] - \omega\psi\mathbb{E}[\xi_{t+1}]$$

Use the Leverage curve B.13 to eliminate $\mathbb{E}[l_{t+1}]$, and use the shock process $\xi_t = \rho_\xi \xi_{t-1} + \varepsilon_{\xi t}$ to eliminate $\mathbb{E}[\xi_{t+1}]$:

$$\begin{aligned}
\mathbb{E}[c_{t+1}] = &\left(1 - \omega(1-\psi)\frac{\sigma-1}{\zeta}\right)\mathbb{E}[x_{t+1}] \\
&+ \omega(1-\psi)\left(1 - \frac{\psi}{\zeta}\right) l_t \\
&- \omega\left((1-\psi)\frac{(1+\sigma\omega(1-\rho_\xi))\psi}{\zeta} + \rho_\xi\psi\right)\xi_t \\
&+ \omega(1-\psi)\frac{\sigma-1}{\zeta}x_t
\end{aligned}$$

Substituting these expressions into the Euler condition yields

$$\begin{aligned}
&x_t + \omega(1-\psi)l_t - \omega\psi\xi_t \\
&\quad = \left(1 - \omega(1-\psi)\frac{\sigma-1}{\zeta}\right)\mathbb{E}[x_{t+1}] \\
&\qquad + \omega(1-\psi)\left(1 - \frac{\psi}{\zeta}\right) l_t \\
&\qquad - \omega\left((1-\psi)\frac{(1+\sigma\omega(1-\rho_\xi))\psi}{\zeta} + \rho_\xi\psi\right)\xi_t \\
&\qquad + \omega(1-\psi)\frac{\sigma-1}{\zeta}x_t - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}])
\end{aligned}$$

which simplifies to yield the following IS curve:

$$x_t = \mathbb{E}[x_{t+1}] - \frac{\zeta}{\zeta+\sigma-1}(i_t - \mathbb{E}_t[\pi_{t+1}]) - \frac{\sigma\omega\psi(1-\psi)}{\zeta+\sigma-1}l_t - \frac{\sigma\omega\psi(\rho_\xi-\psi)}{\zeta+\sigma-1}\xi_t$$

C Welfare criterion

C.1 Helpful log-quadratic approximations of structural relationships

For the aggregate expenditure and financial friction relationships, we derive log-quadratic approximations, which substitute into some log-linear terms in our welfare criteria. We require log-quadratic approximations of the aggregate expenditure relationship in order to appropriately capture welfare costs resulting from fluctuations in the distribution of consumption. We also require log-quadratic approximations of financial relationships, which are not well approximated by log-linear relationships.

Aggregate expenditure The aggregate expenditure relationship

$$X = C + C^e$$

permits the following log-quadratic approximation

$$x_t + \frac{1}{2}x_t^2 = \frac{\bar{c}}{\bar{x}}\left(c_t + \frac{1}{2}c_t^2\right) + \frac{\bar{c^e}}{\bar{x}}\left(c_t^e + \frac{1}{2}c_t^{e2}\right) + \mathcal{O}(z^3) \tag{C.1}$$

Entrepreneurial consumption Combining B.6 with A.5 yields

$$c_t^e = x_t + \psi\left(\frac{1}{2}l^2 + \xi + \frac{1}{2}\xi^2 + l\xi\right) - (1-\psi)l + \mathcal{O}(z^3) \tag{C.2}$$

Worker household consumption. From C.1 and C.2, we can derive the following second order approximation of household consumption

$$\begin{aligned} c_t &= (1+\omega)\left(x_t + \frac{1}{2}x_t^2\right) - \omega\left(c_t^e + \frac{1}{2}c_t^{e2}\right) - \frac{1}{2}\left((1+\omega)x_t - \omega c_t^e\right)^2 + \mathcal{O}(z^3) \\ &= x_t - \omega(\rho_t - l_t) + \frac{1}{2}(1+\omega)x_t^2 - \frac{1}{2}\omega(x_t + \rho_t - l_t)^2 \\ &\quad - \frac{1}{2}\left(x_t - \omega(\rho_t - l_t)\right)^2 + \mathcal{O}(z^3) \end{aligned}$$

Simplifying, with the help of (A.5), yields

$$
\begin{aligned}
c_t &= x_t - \omega(\rho - l) - \frac{1}{2}\omega(1+\omega)(\rho - l)^2 + \mathcal{O}(z^3) \\
&= x_t - \omega\psi\left(\frac{1}{2}l^2 + \xi + \frac{1}{2}\xi^2 + l\xi\right) + \omega(1-\psi)l - \frac{1}{2}\omega(1+\omega)(\rho - l)^2 + \mathcal{O}(z^3) \\
&= x_t - \omega\psi\left(\frac{1}{2}l^2 + \xi + \frac{1}{2}\xi^2 + l\xi\right) + \omega(1-\psi)l \\
&\qquad - \frac{1}{2}\omega(1+\omega)\left(\psi^2\xi^2 - 2\psi(1-\psi)l\xi + (1-\psi)^2 l^2\right) + \mathcal{O}(z^3)
\end{aligned}
$$

It is helpful to write this as

$$
c_t = (1+\omega)x_t - \omega c_t^e - \frac{1}{2}\omega(1+\omega)\left(\psi^2\xi^2 - 2\psi(1-\psi)l\xi + (1-\psi)^2 l^2\right) + \mathcal{O}(z^3)
$$

as the terms $(1+\omega)x_t - \omega c_t$ will drop out of future welfare calculations.

$$
\begin{aligned}
c_t^2 &= \left(x_t - \omega\psi\left(\frac{1}{2}l^2 + \xi + \frac{1}{2}\xi^2 + l\xi\right) + \omega(1-\psi)l\right)^2 + \mathcal{O}(z^3) \\
&= \left(x_t - \omega\psi\xi + \omega(1-\psi)l\right)^2 + \mathcal{O}(z^3) \\
&= x^2 - 2\omega\psi x\xi + 2\omega(1-\psi)xl + \omega^2\psi^2\xi^2 - 2\omega^2\psi(1-\psi)l\xi + \omega^2(1-\psi)^2 l^2 + \mathcal{O}(z^3)
\end{aligned}
$$

C.2 The worker household

The household's welfare is

$$
\begin{aligned}
\mathbb{V} &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{\tau}\left[u(C_t) - v(N_t)\right] \\
&= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{\tau}\left[\begin{array}{c} \left(\frac{C_t - C^*}{C^*}\right) C^* u'(C^*) + \frac{1}{2}\left(\frac{C_t - C^*}{C^*}\right)^2 C^{*2} u''(C^*) \\ -\left(\frac{N_t - N^*}{N^*}\right) N^* v'(N^*) - \frac{1}{2}\left(\frac{N_t - N^*}{N^*}\right)^2 N^* v''(N^*) \end{array}\right] + k
\end{aligned}
$$

$$\mathbb{V} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{\tau} C^* u'(C^*) \left[\begin{array}{l} \left(\frac{C_t - C^*}{C^*}\right) - \frac{\sigma}{2}\left(\frac{C_t - C^*}{C^*}\right)^2 \\ -\frac{N^* v'(N^*)}{C^* u'(C^*)} \left[\left(\frac{N_t - N^*}{N^*}\right) + \frac{\varphi}{2}\left(\frac{N_t - N^*}{N^*}\right)^2 \right] \end{array} \right] + k'$$

$$= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{\tau} C^* u'(C^*) \left[\begin{array}{l} \left(\frac{C_t - C^*}{C^*}\right) - \frac{\sigma}{2}\left(\frac{C_t - C^*}{C^*}\right)^2 \\ -\frac{N^* Y^*}{C^* N^*}(1-\alpha) \left[\left(\frac{N_t - N^*}{N^*}\right) + \frac{\varphi}{2}\left(\frac{N_t - N^*}{N^*}\right)^2 \right] \end{array} \right] + k'$$

$$\mathbb{V} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{\tau} C^* u'(C^*) \left[\left(c_t + \tfrac{1}{2}c_t^2\right) - \tfrac{\sigma}{2}c_t^2 - (1+\omega)(1-\alpha)\left[\left(n_t + \tfrac{1}{2}n_t^2\right) + \tfrac{\varphi}{2}n_t^2\right] \right] + k'$$

where the term $(1+\omega)$ reflects the fact that the household consumption share of output is $1/(1+\omega)$.

We can express the above value function in terms of a within-period loss function:

$$\begin{aligned}
\mathbb{L} &= -c_t + \frac{\sigma-1}{2}c_t^2 + (1+\omega)(1-\alpha)\left(n_t + \frac{1+\varphi}{2}n_t^2\right) \\
&= -c_t + \frac{\sigma-1}{2}c_t^2 + (1+\omega)\frac{1}{2}\frac{1+\varphi}{1-\alpha}(x_t - a_t)^2 \\
&= -(1+\omega)x_t + \omega c_t^e + \frac{1}{2}\omega(1+\omega)\left((1-\psi)^2 l^2 - 2\psi(1-\psi)l\xi\right) \\
&\quad + \frac{\sigma-1}{2}\left(x^2 - 2\omega\psi x\xi + 2\omega(1-\psi)xl - 2\omega^2\psi(1-\psi)l\xi + \omega^2(1-\psi)^2 l^2\right) \\
&\quad + \frac{1}{2}\frac{(1+\omega)(1+\varphi)}{1-\alpha}(x_t^2 - 2x_t a_t) \\
&= \frac{1}{2}\left(\sigma - 1 + (1+\omega)\chi\right)x_t^2 - (1+\omega)\chi x_t a_t \\
&\quad + \frac{1}{2}\omega\left(1+\sigma\omega\right)(1-\psi)\left((1-\psi)l^2 - 2\psi l\xi\right) \\
&\quad + \omega(\sigma-1)\left((1-\psi)xl - \psi x\xi\right) \\
&\quad - (1+\omega)x_t + \omega c_t^e
\end{aligned}$$

C.3 Entrepreneurs

The lifetime value of an individual entrepreneur can be expressed as follows:

$$\mathbb{V}^e = \mathbb{E}\sum_{t=0}^{\infty}(\beta^e)^t \log C_t^e(\theta^t)$$

where θ^t captures the history of idiosyncratic shocks realised by the individual entrepreneur. We decompose this sum into the time zero consumption and the sequence of consumption growth over time, for some entrepreneur who at time zero receives mean consumption $\bar{c}_0^e$

$$(1-\beta^e)\mathbb{V}^e = \log \bar{c}_0^e + \mathbb{E}\sum_{t=0}^{\infty}(\beta^e)^t \log \frac{C_t^e(\theta^t)}{C_{t-1}^e(\theta^{t-1})}$$

Consumption growth can be further decomposed into the aggregate mean consumption growth across all entrepreneurs, and the idiosyncratic component.

$$(1-\beta^e)\mathbb{V}^e = \log \bar{C}_0^e + \mathbb{E}\sum_{t=0}^{\infty}(\beta^e)^t \log \frac{\bar{C}_t^e}{\bar{C}_{t-1}^e} g(\xi_t, l_t)$$

where $g(\xi_t, l_t)$ captures the welfare costs of the idiosyncratic component of growth in net wealth for some entrepreneur with time t aggregate states ξ_t, l_t. We can write

$$\mathbb{V}^e = \frac{1}{1-\beta^e}\log \bar{C}_0^e + \mathbb{E}\sum_{t=0}^{\infty}(\beta^e)^t \log \bar{C}_t^e + \frac{1}{1-\beta^e}\mathbb{E}\sum_{t=0}^{\infty}(\beta^e)^t \log g(\xi_t, l_t)$$

which permits a second order loss function approximation

$$\mathbb{L}^e = \frac{\kappa_{\xi\xi}}{2}\text{var}(\xi_t) + \frac{\kappa_{ll}}{2}\text{var}(l_t) + \kappa_{l\xi}\text{cov}(l_t, \xi_t)$$

$$\mathbb{L}^e = \frac{1}{2}\left[\kappa_{ll}\text{var}(l_t) + 2\kappa_{l\xi}\text{cov}(l_t, \xi_t)\right] + \text{t.i.p}$$

where, without loss of generality, $\kappa_{l\xi} := \dfrac{g_{l\xi}g - g_l g_\xi}{g^2(1-\beta^e)}$, a measure of the convexity of value costs of productive risk and leverage, holding mean consumption growth constant.

C.4 Aggregate welfare

Following Negishi (1960), we can derive the Pareto weights that are consistent with the competitive equilibrium allocations being those resulting of a policymaker's optimal initial wealth allocation. By applying the resulting Negishi weights to our worker and entrepreneurs' loss functions, we remove any redistribution motive from our monetary policy and macroprudential policy analysis. Any resulting welfare gains from optimal policy can be interpreted as increases in the efficiency of allocations.

The policymaker's loss function can be described as follows:

$$\Lambda = 2(\omega L^e + L)$$

where ω, the ratio of steady state entrepreneurial consumption to worker consumption, is equal to the ratio of the Negishi weights attached to the entrepreneurs' and workers' utility respectively.

$$\begin{aligned}\Lambda = &\frac{1}{2}\left(\sigma - 1 + (1+\omega)\chi\right)x_t^2 - (1+\omega)\chi x_t a_t \\ &+ \frac{1}{2}\omega\left(1+\sigma\omega\right)(1-\psi)\left((1-\psi)l^2 - 2\psi l\xi\right) \\ &+ \omega(\sigma-1)\left((1-\psi)xl - \psi x\xi\right) \\ &- (1+\omega)x_t + \omega c_t^e \\ &+ \omega\kappa_{ll}l_t^2 + 2\omega\kappa_{l\xi}l_t\xi_t + (1+\omega)\frac{\varepsilon}{\lambda}\pi_t^2 + \text{t.i.p.}\end{aligned}$$

Collecting like terms,

$$\begin{aligned}\Lambda = &\frac{1}{2}(1+\omega)\frac{\varepsilon}{\lambda}\pi_t^2 + \frac{1}{2}\left(\sigma - 1 + (1+\omega)\chi\right)x_t^2 - (1+\omega)\chi x_t a_t \\ &+ \frac{1}{2}\omega\left((\zeta-\psi)(1-\psi) + \kappa_{ll}\right)l^2 - \omega\left((\zeta-\psi)\psi - \kappa_{l\xi}\right)l\xi \\ &+ \omega(\sigma-1)\left((1-\psi)xl - \psi x\xi\right) + \text{t.i.p.}\end{aligned} \tag{1.8}$$

C.5 The monetary policymaker's objective under log utility $\sigma = 1$

Under log utility, l_t^2 and $l_t \xi_t$ become independent of monetary policy (they remain dependent on policy under the macroprudential policymaker's problem). The term $(\sigma - 1)(2\omega(1-\psi)x_t l_t - 2\omega\gamma x_t \xi_t)$ captures the welfare effects of fluctuations in the distribution of consumption. When both agents have the same preferences over consumption, distributional fluctuations have no effect on social welfare at the margin. After simplifying, we're left with

$$\Lambda = \frac{1}{2}(1+\omega)\left(\frac{\varepsilon}{\lambda}\pi_t^2 + \chi x_t^2 - 2\chi x_t a_t\right) + \text{t.i.p},$$

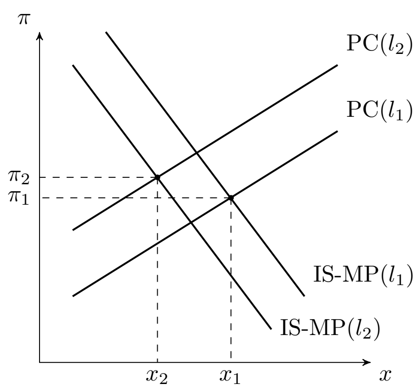
which is identical, up to scaling and terms independent of policy, to the loss function of the policymaker under log utility in the standard New Keynesian model (see for example Galí, 2008, Ch.4 Appendix). Terms independent of monetary policy remain important for welfare, and achieving $\Lambda = 0$ $(+\text{t.i.p})$ in all periods does not imply first best efficiency or even second best constrained efficiency.

D Deriving the NK Curve

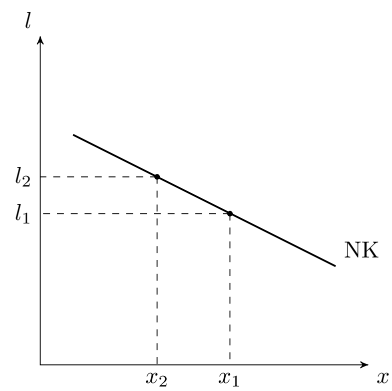
To derive the NK curve relation, we first combine the IS curve and the interest rate policy to obtain a contemporaneous direct relation between output and inflation, herein called IS-MP. Equations (1.1) and the Taylor rule $i_t = \phi_\pi \pi_t$ imply the following,

$$\pi_t = \frac{1}{\phi_\pi}\mathbb{E}_t[\pi_{t+1}] + \frac{\zeta+\sigma-1}{\zeta\phi_\pi}(\mathbb{E}_t[x_{t+1}] - x_t) - \frac{(\zeta-1)\psi}{\zeta\phi_\pi}l_t \tag{D.1}$$

Equation (D.1) presents a negative relation between current period inflation π_t and output x_t, holding all else equal. The Phillips curve, on the other hand, presents a positive relation between the variables. Note also that an changes in leverage have two simultaneous impact in the diagram. Following the IS-MP schedule, an increase (decrease) in leverage decreases (increases) inflation for every level of output. Whereas in the Phillips curve what happens is the opposite. Graphically, this means that following an increase in leverage, the PC shifts upward, while the IS-MP curve shifts downward. Plotting both schedules and the aforementioned leverage impact we obtain the graphic derivation of the NK curve as follows:



(a) IS-MP-PC diagram



(b) NK Curve

Figure 1: NK curve graphic derivation

D.1 Algebraic derivation of the NK curve under log utility

We first seek a general solution to the following IS-PC system, derived from (1.1) and (1.2) with the interest rate rule $i_t = \phi_\pi \pi_t$:

$$x_t = \mathbb{E}[x_{t+1}] - (\phi_\pi \pi_t - \mathbb{E}_t[\pi_{t+1}]) - \frac{(\zeta - 1)\psi}{\zeta} l_t$$
$$\pi_t = \beta \mathbb{E}_t[\pi_{t+1}] + \lambda\chi x_t + \vartheta_l l_t$$

for bounded process l.

Eliminating x_t,

$$\pi_t - \beta\mathbb{E}_t[\pi_{t+1}] - \vartheta_l l_t = \mathbb{E}_t[\pi_{t+1} - \beta\pi_{t+2} - \vartheta_l l_{t+1}] - \lambda\chi(\phi_\pi\pi_t - \mathbb{E}_t[\pi_{t+1}]) - \lambda\chi\frac{(\zeta-1)\psi}{\zeta} l_t$$

The deterministic component is

$$0 = -\pi_t + \beta\pi_{t+1} + \pi_{t+1} - \beta\pi_{t+2} - \lambda\chi\phi_\pi\pi_t + \lambda\chi\pi_{t+1}$$

with characteristic equation

$$0 = (1 + \lambda\chi\phi_\pi) - (1 + \beta + \lambda\chi)\phi + \beta\phi^2$$

and eigenvalues

$$\varphi = \frac{(1 + \beta + \lambda\chi) \pm \sqrt{(1 + \beta + \lambda\chi)^2 - 4\beta(1 + \lambda\chi\phi_\pi)}}{2\beta}$$

Guess the solution

$$\pi_t = \varphi_1 \pi_{t-1} + \mu l_t,$$

where φ_1 is the stable eigenvalue.

$$\pi_t - \beta\mathbb{E}_t[\pi_{t+1}] - \vartheta_l l_t = \mathbb{E}_t[\pi_{t+1} - \beta\pi_{t+2} - \vartheta_l l_{t+1}] - \lambda\chi(\phi_\pi\pi_t - \mathbb{E}_t[\pi_{t+1}]) - \lambda\chi\frac{(\zeta-1)\psi}{\zeta} l_t$$

$$(1+\lambda\chi\phi_\pi)\pi_t - (1+\beta+\lambda\chi)\mathbb{E}_t[\pi_{t+1}] + \beta\mathbb{E}_t[\pi_{t+2}] = \vartheta_l l_t - \vartheta_l\mathbb{E}_t[l_{t+1}] - \lambda\chi\frac{(\zeta-1)\psi}{\zeta} l_t$$

$$(1+\lambda\chi\phi_\pi)\pi_t - (1+\beta+\lambda\chi)\mathbb{E}_t[\pi_{t+1}] + \beta\mathbb{E}_t[\pi_{t+2}] = (1-\phi_1)\vartheta_l l_t - \lambda\chi\frac{(\zeta-1)\psi}{\zeta}l_t$$

where ϕ_1 is the persistence of the process for l_t. Solving for μ we have

$$-\frac{(1+\lambda\chi\phi_\pi)}{\varphi_1}\mu\mathbb{E}_t[l_{t+1}] + \beta\mu\mathbb{E}_t[l_{t+2}] = (1-\phi_1)\vartheta_l l_t - \lambda\chi\frac{(\zeta-1)\psi}{\zeta}l_t,$$

and ultimately

$$\mu = \frac{\varphi_1}{\phi_1}\frac{\lambda\chi\frac{(\zeta-1)\psi}{\zeta} - (1-\phi_1)\vartheta_l}{1+\lambda\chi\phi_\pi - \beta\varphi_1\phi_1}.$$

Substituting the solution for π_t into the Phillips curve yields

$$\pi_t = \beta\varphi_1\pi_t + \beta\mu\phi_1 l_t + \lambda\chi x_t + \vartheta_l l_t$$

$$\lambda\chi x_t = (1-\beta\varphi_1)\pi_t - (\vartheta_l + \beta\mu\phi_1)l_t$$

Subtracting $\varphi_1\lambda\chi x_{t-1}$ from both sides,

$$\lambda\chi x_t - \varphi_1\lambda\chi x_{t-1} = (1-\beta\varphi_1)\pi_t - \varphi_1(1-\beta\varphi_1)\pi_{t-1} - (\vartheta_l+\beta\mu\phi_1)l_t + \varphi_1(\vartheta_l+\beta\mu\phi_1)l_{t-1}$$

$$\lambda\chi x_t = \varphi_1\lambda\chi x_{t-1} - (\vartheta_l + \mu(\beta(\phi_1+\varphi_1)-1))l_t + \varphi_1(\vartheta_l+\beta\mu\phi_1)l_{t-1}$$

Current output is decreasing in current period leverage, which gives us the NK curve. In the limit, as the interest rate response to current inflation π_π increases, the relationship between output and leverage steepens towards the flexible price relationship $\lambda\chi x_t = -(\vartheta_l + \mu(\beta(\phi_1+\varphi_1)-1))l_t$.

E The flexible price model

E.1 The flexible price model in full

The following equations carry over from the sticky price model
Risk Sharing

$$\sigma c_t - c_t^e = \sigma c_{t-1} - c_{t-1}^e - \rho_t \tag{B.3}$$

Aggregate Demand

$$x_t = \frac{\bar{c}}{\bar{x}} c_t + \frac{\bar{c}^e}{\bar{x}} c_t^e \tag{B.4}$$

Risk premia

$$\rho_t = \frac{LT}{1+LT} l_t + \frac{L}{1+LT} \tau_t \tag{B.5}$$

Leverage

$$x_t = c_t^e - \rho_t + l_t \tag{B.6}$$

Wedge

$$\tau_t = \theta_l l_t + \theta_\xi \xi_t \tag{B.8}$$

The production and labour market equations are as follows:
Production

$$x_t = a_t + (1-\alpha) n_t \tag{E.1}$$

Labour supply

$$-\sigma c_t = \varphi n_t - w_t \tag{E.2}$$

Labour demand

$$w_t = x_t - n_t - \tau_t \tag{E.3}$$

E.2 Equilibrium production

Let $\chi = \dfrac{1+\varphi}{1-\alpha}$. From Equations E.1, E.2, E.3, we can derive the following expression for equilibrium output

$$(\chi - 1)\, x_t = \chi a_t - \sigma c_t - \tau_t$$

Now, we use (B.8) and (B.14), we can eliminate c and τ

$$(\chi - 1)x_t = \chi a_t - \sigma\left(x - \omega(\rho - l)\right) - (\theta_l l_t + \theta_\xi \xi_t)$$

Simplifying yields

$$(\chi - 1)x_t = \chi a_t - \sigma x + \sigma\omega(\psi\xi - (1-\psi)l) - (\theta_l l_t + \theta_\xi \xi_t)$$

$$(\chi + \sigma - 1)x_t = \chi a_t - (\zeta - 1 + \theta_l)l_t - \left(\theta_\xi - \sigma\omega\psi\right)\xi_t \qquad (3.1)$$

Equations 1.3 and 3.1 describe a two equation solved flexible price model.

E.3 Dynamics

From the system described by (1.3) and (3.1), we can solve for output in terms of shocks and past values of output:

$$\begin{aligned}
&\frac{\zeta}{\zeta - 1 + \theta_l}\left(\chi a_t - (\chi + \sigma - 1)x_t - \left(\theta_\xi - \sigma\omega\psi\right)\xi_t\right) \\
&\qquad = \frac{\zeta - \psi}{\zeta - 1 + \theta_l}\left(\chi a_{t-1} - (\chi + \sigma - 1)x_{t-1} - \left(\theta_\xi - \sigma\omega\psi\right)\xi_{t-1}\right) \\
&\qquad\quad + \sigma\omega\psi\xi_t - (1 + \sigma\omega)\psi\xi_{t-1} - (\sigma - 1)(x_t - x_{t-1}).
\end{aligned}$$

Simplifying yields

$$\begin{aligned}
&\left(\zeta\chi - (\theta_l - 1)(\sigma - 1)\right)x_t \\
&\quad = \zeta\chi\varepsilon_{at} - \left(\zeta\theta_\xi + (\theta_l - 1)\sigma\omega\psi\right)\varepsilon_{\xi t} \\
&\qquad + \left((\zeta - \psi)\chi - (\psi + \theta_l - 1)(\sigma - 1)\right)x_{t-1} - \left(\zeta(1 - \rho_a) - \psi\right)\chi a_{t-1} \\
&\qquad + \left[\left(\zeta\theta_\xi + (\theta_l - 1)\sigma\omega\psi\right)(1 - \rho_\xi) + \psi((\zeta + \theta_l - 1) - (\theta_\xi - \sigma\omega\psi))\right]\xi_{t-1}.
\end{aligned} \qquad (E.4)$$

F Derivations for Section 3

F.1 Optimal policy under flexible prices

Our loss function is derived from Equation 1.8, with $\sigma = 1$,

$$\Lambda = \frac{1}{2}(1+\omega)\chi\left(x_t^2 - 2x_t a_t\right) \\ + \frac{1}{2}\omega\left(\kappa_{ll} + (\zeta-\psi)(1-\psi)\right)l_t^2 + \omega\left(\kappa_{l\xi} - (\zeta-\psi)\psi\right)l_t\xi_t + \text{t.i.p.}$$

From 3.1, we have the following aggregate supply condition

$$\chi(x_t - a_t) = -(\zeta + \theta_l - 1)l_t - \left(\theta_\xi - \sigma\omega\psi\right)\xi_t$$

The macroprudential policymaker's problem can be expressed as a Lagrangian:

$$\mathcal{L} = \mathbb{E}\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{2}\left[(1+\omega)\chi\left(x_t^2 - 2x_t a_t\right) + \omega\hat{\kappa}_{ll}l_t^2 + 2\omega\hat{\kappa}_{l\xi}l_t\xi_t\right]\right. \\ - \mu_t\left[\chi x_t - \chi a_t + (\zeta+\theta_l-1)l_t + \left(\theta_\xi - \sigma\omega\psi\right)\xi_t\right] \\ \left. -\nu_t\left[\zeta l_{t+1} - (\zeta-\psi)l_t - \omega\psi\xi_{t+1} + (1+\omega)\psi\xi_t\right]\right\}.$$

where

$$\hat{\kappa}_{ll} = \kappa_{ll} + (\zeta-\psi)(1-\psi)$$
$$\hat{\kappa}_{l\xi} = \kappa_{l\xi} - (\zeta-\psi)\psi$$

The first order conditions are

$$x_t: \quad 0 = (1+\omega)\chi(x_t - a_t) - \chi\mu_t$$
$$l_t: \quad 0 = \omega\hat{\kappa}_{ll}l_t + \omega\hat{\kappa}_{l\xi}\xi_t - (\zeta+\theta_l-1)\mu_t + (\zeta-\psi)\nu_t - \frac{\zeta}{\beta}\nu_{t-1}$$

Using the aggregate supply condition to eliminate x_t yields

$$(\zeta+\theta_l-1)l_t + \left(\theta_\xi - \sigma\omega\psi\right)\xi_t = -\frac{\chi}{1+\omega}\mu_t.$$

Eliminating μ_t,

$$0 = \omega\left(\kappa_{ll} + (\zeta-\psi)(1-\psi)\right) l_t + \omega\left(\kappa_{l\xi} - (\zeta-\psi)\psi\right)\xi_t + \frac{1+\omega}{\chi}(\zeta+\theta_l-1)\left((\zeta+\theta_l-1)l_t + (\theta_\xi - \sigma\omega\psi)\,\xi_t\right) + (\zeta-\psi)\nu_t - \frac{\zeta}{\beta}\nu_{t-1}$$

$$\left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2\right) l_t = -\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi\right)\xi_t - \chi(\zeta-\psi)\nu_t + \chi\frac{\zeta}{\beta}\nu_{t-1} \tag{F.1}$$

Substituting (F.1) into the Leverage curve yields

$$\begin{aligned} 0 = \zeta &\left[-\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi\right)\mathbb{E}_t[\xi_{t+1}] - \chi(\zeta-\psi)\mathbb{E}_t[\nu_{t+1}] + \chi\frac{\zeta}{\beta}\nu_t\right] \\ &- (\zeta-\psi)\left[-\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi\right)\xi_t - \chi(\zeta-\psi)\nu_t + \chi\frac{\zeta}{\beta}\nu_{t-1}\right] \\ &- \left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2\right)\omega\psi\xi_{t+1} + \left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2\right)(1+\omega)\psi\xi_t. \end{aligned}$$

Dropping shock terms:

$$\begin{aligned} 0 = \zeta &\left[-\chi(\zeta-\psi)\mathbb{E}_t[\nu_{t+1}] + \chi\frac{\zeta}{\beta}\nu_t\right] \\ &- (\zeta-\psi)\left[-\chi(\zeta-\psi)\nu_t + \chi\frac{\zeta}{\beta}\nu_{t-1}\right] \end{aligned}$$

Simplifying yields

$$\beta\zeta\left(\zeta-\psi\right)\mathbb{E}_t[\nu_{t+1}] - \left(\zeta^2 + \beta\left(\zeta-\psi\right)^2\right)\nu_t + \zeta\left(\zeta-\psi\right)\nu_{t-1} = 0$$

The characteristic equation is

$$\beta\zeta\left(\zeta-\psi\right)\phi^2 - \left(\zeta^2 + \beta\left(\zeta-\psi\right)^2\right)\phi + \zeta\left(\zeta-\psi\right) = 0$$

with solutions

$$\phi_1 = \frac{\zeta-\psi}{\zeta}, \qquad \phi_2 = \frac{\zeta}{\beta\left(\zeta-\psi\right)}$$

The first solution, ϕ_1, is inside the unit circle, giving us the general stable solution below:

$$v_t = \frac{\zeta-\psi}{\zeta} v_{t-1} + \eta\xi_t \tag{F.2}$$

Substituting (F.2) into the leverage constraint allows us to solve for η.

$$\begin{aligned}
0 &= \zeta\left[-\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi\right)\mathbb{E}_t[\xi_{t+1}] - \chi(\zeta-\psi)\mathbb{E}_t[\nu_{t+1}] + \chi\frac{\zeta}{\beta}\nu_t\right] \\
&\quad - (\zeta-\psi)\left[-\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi\right)\xi_t - \chi(\zeta-\psi)\nu_t + \chi\frac{\zeta}{\beta}\nu_{t-1}\right] \\
&\quad - \left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2\right)\omega\psi\mathbb{E}_t[\xi_{t+1}] + \left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2\right)(1+\omega)\psi\xi_t. \\
0 &= \zeta\left[-\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi\right)\rho_\xi\xi_t - \chi(\zeta-\psi)\eta\rho_\xi\xi_t + \chi\frac{\zeta}{\beta}\eta\xi_t\right] \\
&\quad - (\zeta-\psi)\left[-\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi\right)\xi_t\right] \\
&\quad - \left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2\right)\omega\psi\rho_\xi\xi_t + \left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2\right)(1+\omega)\psi\xi_t. \\
0 &= \eta\chi\zeta\left[\zeta(1-\beta\rho_\xi) + \beta\rho_\xi\psi\right] + \beta\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi\right)\left(\zeta(1-\rho_\xi) - \psi\right) \\
&\quad + \beta\left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2\right)(1+\omega(1-\rho_\xi))\psi.
\end{aligned}$$

$$\eta = -\frac{\beta}{\chi\zeta}\frac{\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi\right)\left(\zeta(1-\rho_\xi) - \psi\right) + \left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2\right)(1+\omega(1-\rho_\xi))\psi}{\zeta(1-\beta\rho_\xi) + \beta\rho_\xi\psi} \tag{F.3}$$

Now, write the Leverage curve as follows:

$$0 = \zeta l_{t+1} - (\zeta-\psi)\, l_t - \omega\psi\xi_{t+1} + (1+\omega)\psi\xi_t + \delta'_{t+1}$$

where δ'_{t+1} is an expectation error shock. Use (F.1) to eliminate l,

$$\begin{aligned}
0 &= \zeta\left[-\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi\right)\xi_{t+1} - \chi(\zeta-\psi)\nu_{t+1} + \chi\frac{\zeta}{\beta}\nu_t\right] \\
&\quad - (\zeta-\psi)\left[-\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi\right)\xi_t - \chi(\zeta-\psi)\nu_t + \chi\frac{\zeta}{\beta}\nu_{t-1}\right] \\
&\quad + \left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2\right)\left(-\omega\psi\xi_{t+1} + (1+\omega)\psi\xi_t + \delta'_{t+1}\right)
\end{aligned}$$

and use F.2 to eliminate ν,

$$
\begin{aligned}
0 = \zeta & \left[-\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi \right)\xi_{t+1} - \chi(\zeta-\psi)\eta\xi_{t+1} + \chi\frac{\zeta}{\beta}\eta\xi_t \right] \\
& - (\zeta-\psi)\left[-\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi \right)\xi_t \right] \\
& + \left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2 \right)\left(-\omega\psi\xi_{t+1} + (1+\omega)\psi\xi_t + \delta'_{t+1} \right)
\end{aligned}
$$

retain only terms measurable in $t+1$,

$$
\begin{aligned}
0 = \zeta & \left[-\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi \right)\varepsilon_{\xi,t+1} - \chi(\zeta-\psi)\eta\varepsilon_{\xi,t+1} \right] \\
& + \left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2 \right)\left(-\omega\psi\varepsilon_{\xi,t+1} + \delta'_{t+1} \right)
\end{aligned}
$$

$$
\delta'_{t+1} = \left(\zeta \left[\frac{\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi \right) + \chi(\zeta-\psi)\eta}{\left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2 \right)} \right] + \omega\psi \right)\varepsilon_{\xi,t+1}
$$

Now use (F.3) to eliminate η,

$$
\begin{aligned}
\delta'_{t+1} = \Bigg(& \zeta \left[\frac{\left(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi \right) - \chi(\zeta-\psi)\frac{\beta}{\chi\zeta}\frac{(\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi)(\zeta(1-\rho_\xi)-\psi)}{\zeta(1-\beta\rho_\xi)+\beta\rho_\xi\psi}}{\left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2 \right)} \right] \\
& -\zeta \left[\frac{\chi(\zeta-\psi)\frac{\beta}{\chi\zeta}\frac{\left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2 \right)(1+\omega(1-\rho_\xi))\psi}{\zeta(1-\beta\rho_\xi)+\beta\rho_\xi\psi}}{\left(\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2 \right)} \right] + \omega\psi \Bigg)\varepsilon_{\xi,t+1}
\end{aligned}
$$

$$
\delta'_{t+1} = \left(\frac{\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi}{\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2}\left(\frac{\zeta^2 - \beta(\zeta-\psi)^2}{\zeta - (\zeta-\psi)\beta\rho_\xi} \right) - \frac{\beta(\zeta-\psi)(1+\omega(1-\rho_\xi))\psi}{\zeta - (\zeta-\psi)\beta\rho_\xi} + \omega\psi \right)\varepsilon_{\xi,t+1}
$$

$$
\delta'_{t+1} = \left(\frac{\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi}{\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2}\left(\frac{\zeta^2 - \beta(\zeta-\psi)^2}{\zeta - (\zeta-\psi)\beta\rho_\xi} \right) - \frac{(1+\omega)\beta(\zeta-\psi) - \omega\zeta}{\zeta - (\zeta-\psi)\beta\rho_\xi}\,\psi \right)\varepsilon_{\xi,t+1}
$$

The ratio

$$
\frac{\chi\omega\hat{\kappa}_{l\xi} + (1+\omega)\vartheta_l\vartheta_\xi}{\chi\omega\hat{\kappa}_{ll} + (1+\omega)\vartheta_l^2}
$$

is the current period marginal rate of transformation between the social costs of uncertainty and the social costs of leverage. It tells the policymaker how much leverage must fall in order to offset the social costs of an increase in uncertainty.

In the competitive equilibrium, uncertainty shocks increase current period leverage but they reduce leverage over longer time horizons. When uncertainty is high, the return to inside wealth is also high, and entrepreneurs' inside wealth grows quickly. As leverage is persistent, macroprudential policy has a persistent effect on the path of leverage, and can exacerbate the medium term fall in leverage in response to a contractionary uncertainty shock. This persistence may not be desirable. The second term in brackets,

$$-\frac{(1+\omega)\beta(\zeta-\psi)-\omega\zeta}{\zeta-(\zeta-\psi)\beta\rho_\xi}\,\psi,$$

reflects the persistent effect of current period uncertainty on future leverage, and dampens the optimal macroprudential response to uncertainty shocks.

F.2 Joint optimal policy

In this section we solve for jointly optimal monetary and prudential policy under commitment. We separate the problem into two parts. Under log utility, the effect of the monetary policymaker's action on leverage is mediated through the optimal policy of the prudential policymaker. So, we solve for the monetary policymaker's problem (ie. the path of x, π) first, then the prudential policymaker's problem (l).

The combined policymaker's problem is

$$\begin{aligned}\min_{\pi,x,l}\mathbb{E}\sum_{t=0}^{\infty}\beta^t\Bigg\{\frac{1}{2}\left[(1+\omega)\left(\frac{\varepsilon}{\lambda}\pi_t^2+\chi\left(x_t^2-2x_ta_t\right)\right)+\omega\hat{\kappa}_{ll}l_t^2+2\omega\hat{\kappa}_{l\xi}l_t\xi_t\right]\\ -\mu_t\left[-\pi_t+\beta\pi_{t+1}+\lambda\chi x_t-\lambda\chi a_t+\lambda\vartheta_l l_t+\lambda\vartheta_\xi\xi_t\right]\\ -\nu_t\left[\zeta l_{t+1}-(\zeta-\psi)\,l_t-\omega\psi\xi_{t+1}+(1+\omega)\psi\xi_t\right]\Bigg\}.\end{aligned}$$

The first order conditions are

$$
\begin{aligned}
\pi: &\qquad 0 = (1+\omega)\frac{\varepsilon}{\lambda}\pi_t + \mu_t - \mu_{t-1} \\
x: &\qquad 0 = (1+\omega)\chi\,(x_t - a_t) - \mu_t\lambda\chi \\
l: &\qquad 0 = \omega(\hat{\kappa}_{ll} l_t + \hat{\kappa}_{l\xi}\xi_t) - \lambda\vartheta_l\mu_t + (\zeta - \psi)\,\nu_t - \frac{\zeta}{\beta}\nu_{t-1}
\end{aligned}
$$

We first solve for π, x then solve for l. Using the first order conditions to eliminate π, x from the phillips curve,

$$0 = -\pi_t + \beta\mathbb{E}_t[\pi_{t+1}] + \lambda\chi x_t - \lambda\chi a_t + bw_t$$

where $b = [\lambda\vartheta_l \quad \lambda\vartheta_\xi]$, $\quad w_t = [l_t \quad \xi_t]'$. The product bw_t is a bounded process.

$$0 = (\mu_t - \mu_{t-1}) - \beta(\mathbb{E}_t[\mu_{t+1}] - \mu_t) + \lambda\chi\varepsilon\mu_t + \frac{(1+\omega)\varepsilon}{\lambda} bw_t.$$

Simplifying,

$$0 = -\beta\mathbb{E}_t[\mu_{t+1}] + (1+\beta+\lambda\chi\varepsilon)\mu_t - \mu_{t-1} + \frac{(1+\omega)\varepsilon}{\lambda} bw_t.$$

The characteristic equation is

$$0 = \beta\varphi^2 - (1+\beta+\lambda\chi\varepsilon)\varphi + 1,$$

and the stable and unstable roots are given by φ_1, φ_2 respectively:

$$\varphi_1 = \frac{(1+\beta+\lambda\chi\varepsilon) - \sqrt{(1+\beta+\lambda\chi\varepsilon)^2 - 4\beta}}{2\beta}, \qquad \varphi_2 = \frac{(1+\beta+\lambda\chi\varepsilon) + \sqrt{(1+\beta+\lambda\chi\varepsilon)^2 - 4\beta}}{2\beta}.$$

The unique solution is

$$\mu_t = \varphi_1\mu_{t-1} - \frac{(1+\omega)\varepsilon}{\lambda}\beta^{-1}\varphi_2^{-1}\sum_{j=0}^{\infty}\varphi_2^{-j}\mathbb{E}_t[bw_{t+j}]$$

(See Woodford, 2003, Ch. 7 Eqn. 2.7). We denote this solution as

$$\mu_t = \varphi_1 \mu_{t-1} - \frac{(1+\omega)\varepsilon}{\beta\varphi_2\lambda} bMw_t. \tag{F.4}$$

for the linear map

$$M = \left(I - \frac{1}{\varphi_2}A\right)^{-1}, \qquad \text{where} \qquad A = \begin{bmatrix} \frac{\zeta-\psi}{\zeta} & -\frac{(1+\omega(1-\rho_\xi))\psi}{\zeta} \\ 0 & \rho_\xi \end{bmatrix}.$$

The inflation rate satisfies

$$\varepsilon\pi_t = (1-\varphi_1)\tilde{x}_{t-1} + \frac{\varepsilon}{\beta\varphi_2} bMw_t,$$

where $\tilde{x}_t = x_t - a_t$ (See Woodford, 2003, Ch. 7 Eqn. 2.12).

Now, substitute the first order condition for leverage into the deterministic leverage constraint to derive the deterministic component of the dynamic evolution of the shadow cost of the leverage constraint ν:

$$\begin{aligned} 0 &= \zeta l_{t+1} - (\zeta-\psi)\, l_t \\ &= \zeta[\lambda\vartheta_l\mu_{t+1} - (\zeta-\psi)\,\nu_{t+1} + \frac{\zeta}{\beta}\nu_t] - (\zeta-\psi)\,[\lambda\vartheta_l\mu_t - (\zeta-\psi)\,\nu_t + \frac{\zeta}{\beta}\nu_{t-1}] \\ &= \lambda\vartheta_l(\mu_{t+1} - \phi_1\mu_t) - (\zeta-\psi)\,(\nu_{t+1} - \phi_1\nu_t) + \frac{\zeta}{\beta}(\nu_t - \phi_1\nu_{t-1}) \end{aligned}$$

We can derive the deterministic component of μ by iterating (F.4) backwards:[3]

$$\mu_t = -\frac{(1+\omega)\varepsilon}{\beta\varphi_2\lambda}(bM)^1 \sum_{j=0}^{\infty} \varphi_1^j l_{t-j}$$

$$\mu_{t+1} - \phi_1\mu_t = -\frac{(1+\omega)\varepsilon}{\beta\varphi_2\lambda}(bM)^1 \sum_{j=0}^{\infty} \varphi_1^j \underbrace{(l_{t-j} - \phi_1 l_{t-j-1})}_{=0} = 0$$

[3]Note that the backward looking stochastic component of μ depends on the macroprudential policy, but the deterministic component does not.

Therefore we have

$$0 = -(\zeta - \psi)(\nu_{t+1} - \phi_1 \nu_t) + \frac{\zeta}{\beta}(\nu_t - \phi_1 \nu_{t-1})$$

which has the stable solution

$$\nu_t = \phi_1 \nu_{t-1} + u_t,$$

for some process u_t. Now substitute this solution into the leverage constraint

$$\begin{aligned}
0 &= \zeta \mathbb{E}_t[l_{t+1}] - (\zeta - \psi) l_t - \omega\psi \mathbb{E}_t[\xi_{t+1}] + (1+\omega)\psi\xi_t \\
&= \omega\hat{\kappa}_{ll}\mathbb{E}_t[l_{t+1}] - \phi_1\omega\hat{\kappa}_{ll} l_t - \omega\hat{\kappa}_{ll}\omega\frac{\psi}{\zeta}\mathbb{E}_t[\xi_{t+1}] + \omega\hat{\kappa}_{ll}(1+\omega)\frac{\psi}{\zeta}\xi_t \\
&= -\omega\hat{\kappa}_{l\xi}\mathbb{E}_t[\xi_{t+1}] + \lambda\vartheta_l\mathbb{E}_t[\mu_{t+1}] - (\zeta-\psi)\mathbb{E}_t[\nu_{t+1}] + \frac{\zeta}{\beta}\nu_t \\
&\quad - \phi_1[-\omega\hat{\kappa}_{l\xi}\xi_t + \lambda\vartheta_l\mu_t - (\zeta-\psi)\nu_t + \frac{\zeta}{\beta}\nu_{t-1}] \\
&\quad - \omega\hat{\kappa}_{ll}\omega\frac{\psi}{\zeta}\mathbb{E}_t[\xi_{t+1}] + \omega\hat{\kappa}_{ll}(1+\omega)\frac{\psi}{\zeta}\xi_t \\
&= \lambda\vartheta_l(\mathbb{E}_t[\mu_{t+1}] - \phi_1\mu_t) - (\zeta-\psi)(\mathbb{E}_t[\nu_{t+1}] - \phi_1\nu_t) + \frac{\zeta}{\beta}(\nu_t - \phi_1\nu_{t-1}) \\
&\quad + \omega\hat{\kappa}_{ll}(1+\omega(1-\rho_\xi))\frac{\psi}{\zeta}\xi_t + (\phi_1 - \rho_\xi)\omega\hat{\kappa}_{l\xi}\xi_t \\
&= \frac{\beta\lambda\vartheta_l}{\zeta}(\mathbb{E}_t[\mu_{t+1}] - \phi_1\mu_t) - \beta\phi\mathbb{E}_t[u_{t+1}] + u_t + k\xi_t
\end{aligned}$$

$$u_t = \beta\phi\mathbb{E}_t[u_{t+1}] - \frac{\beta\lambda\vartheta_l}{\zeta}(\mathbb{E}_t[\mu_{t+1}] - \phi_1\mu_t) - k\xi_t \tag{F.5}$$

Now, we'll derive a condition for ω, the macroprudential policy action, then attempt to combine it with the restriction above to complete the solution. With macroprudential policy, our (realised) leverage constraint becomes

$$0 = \zeta l_{t+1} - (\zeta - \psi) l_t - \omega\psi\xi_{t+1} + (1+\omega)\psi\xi_t + \delta'_{t+1}$$

where $\mathbb{E}_t[\delta'_{t+1}] = 0$. Substitute in the optimality condition for leverage,

$$
\begin{aligned}
0 &= \zeta l_{t+1} - (\zeta - \psi)\, l_t - \omega\psi\xi_{t+1} + (1+\omega)\psi\xi_t + \delta'_{t+1} \\
&= \omega\hat{\kappa}_{ll} l_{t+1} - \phi_1 \omega\hat{\kappa}_{ll} l_t - \omega\hat{\kappa}_{ll}\omega\frac{\psi}{\zeta}\xi_{t+1} + \omega\hat{\kappa}_{ll}(1+\omega)\frac{\psi}{\zeta}\xi_t + \frac{\omega\hat{\kappa}_{ll}}{\zeta}\delta'_{t+1} \\
&= -\omega\hat{\kappa}_{l\xi}\xi_{t+1} + \lambda\vartheta_l\mu_{t+1} - (\zeta - \psi)\,\nu_{t+1} + \frac{\zeta}{\beta}\nu_t \\
&\quad - \phi_1[-\omega\hat{\kappa}_{l\xi}\xi_t + \lambda\vartheta_l\mu_t - (\zeta - \psi)\,\nu_t + \frac{\zeta}{\beta}\nu_{t-1}] \\
&\quad - \omega\hat{\kappa}_{ll}\omega\frac{\psi}{\zeta}\xi_{t+1} + \omega\hat{\kappa}_{ll}(1+\omega)\frac{\psi}{\zeta}\xi_t + \frac{\omega\hat{\kappa}_{ll}}{\zeta}\delta'_{t+1}
\end{aligned}
$$

Retain only terms measurable in period $t+1$

$$
\begin{aligned}
0 &= \lambda\vartheta_l\,(\mu_{t+1} - \mathbb{E}_t[\mu_{t+1}]) - (\zeta - \psi)\,(\nu_{t+1} - \mathbb{E}_t[\nu_{t+1}]) \\
&\quad - \omega\left(\hat{\kappa}_{l\xi} + \frac{\omega\psi}{\zeta}\hat{\kappa}_{ll}\right)\varepsilon_{\xi t+1} + \frac{\omega\hat{\kappa}_{ll}}{\zeta}\delta'_{t+1} \\
&= \lambda\vartheta_l\,(\mu_{t+1} - \mathbb{E}_t[\mu_{t+1}]) - (\zeta - \psi)\,(u_{t+1} - \mathbb{E}_t[u_{t+1}]) \\
&\quad - \omega\left(\hat{\kappa}_{l\xi} + \frac{\omega\psi}{\zeta}\hat{\kappa}_{ll}\right)\varepsilon_{\xi t+1} + \frac{\omega\hat{\kappa}_{ll}}{\zeta}\delta'_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\omega\hat{\kappa}_{ll}}{\zeta}\delta'_{t+1} &= -\lambda\vartheta_l\,(\mu_{t+1} - \mathbb{E}_t[\mu_{t+1}]) + (\zeta - \psi)\,(u_{t+1} - \mathbb{E}_t[u_{t+1}]) \\
&\quad + \omega\left(\hat{\kappa}_{l\xi} + \frac{\omega\psi}{\zeta}\hat{\kappa}_{ll}\right)\varepsilon_{\xi t+1}
\end{aligned}
\tag{F.6}
$$

Now use (F.5) to solve for the expectation error on the lagrange multiplier attached to the leverage curve. We denote this expectation error by $\Delta^e\mathbb{E}_{t+1}[u_{t+1}] = u_{t+1} - \mathbb{E}_t[u_{t+1}]$, and, without loss of generality, we denote $\Delta^e\mathbb{E}_{t+1}[z_{t+1+j}] := \mathbb{E}_{t+1}[z_{t+1+j}] - \mathbb{E}_t[z_{t+1+j}]$ as the component of the time t expectation error on z_{t+1+j}

that is revealed in period $t+1$.

$$\begin{aligned}
&\Delta^e \mathbb{E}_{t+1}[u_{t+1}] \\
&\quad = -\frac{\beta\lambda\vartheta_l}{\zeta}\Delta^e \mathbb{E}_{t+1}[\mu_{t+2} + \beta\phi_1\mu_{t+3} + (\beta\phi_1)^2\mu_{t+4} + ...] \\
&\qquad + \frac{\beta\lambda\vartheta_l}{\zeta}\phi_1\Delta^e \mathbb{E}_{t+1}[\mu_{t+1} + \beta\phi_1\mu_{t+2} + (\beta\phi_1)^2\mu_{t+3} + ...] \\
&\qquad - k\Delta^e \mathbb{E}_{t+1}[\xi_{t+1} + \beta\phi_1\xi_{t+2} + (\beta\phi_1)^2\xi_{t+3} + ...] \\
&\quad = \frac{\beta\lambda\vartheta_l}{\zeta}\frac{(1+\omega)\varepsilon}{\beta\varphi_2\lambda} bM\left(\phi_2 - \phi_1\right) \\
&\qquad \times \left(\beta\phi_1(\varphi_1 + A) + (\beta\phi_1)^2(\varphi_1^2 + \varphi_1 A + A^2) + (\beta\phi_1)^3(\varphi_1^3 + \varphi_1^2 A + \varphi_1 A^2 + A^3) + ...\right)\Delta^e \mathbb{E}_{t+1}[w_{t+1}] \\
&\qquad - \frac{\beta\lambda\vartheta_l}{\zeta}\phi_1\frac{(1+\omega)\varepsilon}{\beta\varphi_2\lambda} bM\Delta^e \mathbb{E}_{t+1}[w_{t+1}] - \frac{k}{1-\beta\phi_1\rho_\xi}\varepsilon_{\xi t+1}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\left(\beta\phi_1(\varphi_1 + A) + (\beta\phi_1)^2(\varphi_1^2 + \varphi_1 A + A^2) + (\beta\phi_1)^3(\varphi_1^3 + \varphi_1^2 A + \varphi_1 A^2 + A^3) + ...\right) \\
&\quad = -I + \left(\frac{\phi_2}{\phi_2 - \varphi_1}\right)\left(I - \frac{1}{\phi_2}A\right)^{-1},
\end{aligned}$$

which we can use to further simplify the expression,

$$\begin{aligned}
\Delta^e \mathbb{E}_{t+1}[u_{t+1}] &= \frac{\beta\lambda\vartheta_l}{\zeta}\frac{(1+\omega)\varepsilon}{\beta\varphi_2\lambda} bM\left(\phi_2 - \phi_1\right)\left(-I + \left(\frac{\phi_2}{\phi_2 - \varphi_1}\right)\left(I - \frac{1}{\phi_2}A\right)^{-1}\right)\Delta^e \mathbb{E}_{t+1}[w_{t+1}] \\
&\qquad - \frac{\beta\lambda\vartheta_l}{\zeta}\phi_1\frac{(1+\omega)\varepsilon}{\beta\varphi_2\lambda} bM\Delta^e \mathbb{E}_{t+1}[w_{t+1}] - \frac{k}{1-\beta\phi_1\rho_\xi}\varepsilon_{\xi t+1} \\
&= -\frac{\beta\lambda\vartheta_l}{\zeta}\frac{(1+\omega)\varepsilon\phi_2}{\beta\varphi_2\lambda} bM\left(I - \left(\frac{\phi_2 - \phi_1}{\phi_2 - \varphi_1}\right)\left(I - \frac{1}{\phi_2}A\right)^{-1}\right)\Delta^e \mathbb{E}_{t+1}[w_{t+1}] \\
&\qquad - \frac{\phi_2 k}{\phi_2 - \rho_\xi}\varepsilon_{\xi t+1}.
\end{aligned}$$

Or in terms of the price level we have

$$\Delta^e \mathbb{E}_{t+1}[u_{t+1}] = \frac{\beta\lambda\vartheta_l}{\zeta}\Delta^e \mathbb{E}_{t+1}\left[\phi_1\mu_{t+1} - (1-\beta\phi_1^2)\sum_{j=0}^{\infty}(\beta\phi_1)^j\mu_{t+2+j}\right] - \frac{k}{1-\beta\phi_1\rho_\xi}\varepsilon_{\xi t+1}$$

where

$$k := \frac{\beta}{\zeta}\left(\omega\hat{\kappa}_{ll}(1+\omega(1-\rho_\xi))\frac{\psi}{\zeta} + (\phi_1-\rho_\xi)\omega\hat{\kappa}_{l\xi}\right).$$

We can solve for ω by substituting either of the above expressions into F.6. In terms of the price level we have

$$\begin{aligned}
\frac{\omega\hat{\kappa}_{ll}}{\zeta}{\delta'}_{t+1} &= -\lambda\vartheta_l\Delta^e\mathbb{E}_{t+1}[\mu_{t+1}] + (\zeta-\psi)\,\Delta^e\mathbb{E}_{t+1}[u_{t+1}] + \omega\left(\hat{\kappa}_{l\xi} + \frac{\omega\psi}{\zeta}\hat{\kappa}_{ll}\right)\varepsilon_{\xi t+1} \\
&= -\lambda\vartheta_l\Delta^e\mathbb{E}_{t+1}[\mu_{t+1}] + (\zeta-\psi)\,\frac{\beta\lambda\vartheta_l}{\zeta}\Delta^e\mathbb{E}_{t+1}\left(\phi_1\mu_{t+1} - (1-\beta\phi_1^2)\sum_{j=0}^{\infty}(\beta\phi_1)^j\mu_{t+2+j}\right) \\
&\quad + \omega\left(\hat{\kappa}_{l\xi} + \frac{\omega\psi}{\zeta}\hat{\kappa}_{ll}\right)\varepsilon_{\xi t+1} - \frac{\zeta-\psi}{1-\beta\phi_1\rho_\xi}\frac{\beta}{\zeta}\left(\omega\hat{\kappa}_{ll}(1+\omega(1-\rho_\xi))\frac{\psi}{\zeta} + (\phi_1-\rho_\xi)\omega\hat{\kappa}_{l\xi}\right)\varepsilon_{\xi t+1} \\
&= (1+\omega)\varepsilon\vartheta_l(1-\beta\phi_1^2)\Delta^e\mathbb{E}_{t+1}\sum_{j=0}^{\infty}(\beta\phi_1)^j p_{t+1+j} \\
&\quad + \omega\left(\frac{1-\beta\phi_1^2}{1-\beta\phi_1\rho_\xi}\hat{\kappa}_{l\xi} - \frac{\beta\phi_1-\omega(1-\beta\phi_1)}{1-\beta\phi_1\rho_\xi}\frac{\psi}{\zeta}\hat{\kappa}_{ll}\right)\varepsilon_{\xi t+1}
\end{aligned}$$

$$\begin{aligned}
{\delta'}_{t+1} &= \frac{\zeta(1+\omega)\varepsilon\vartheta_l}{\omega\hat{\kappa}_{ll}}(1-\beta\phi_1^2)\Delta^e\mathbb{E}_{t+1}\sum_{j=0}^{\infty}(\beta\phi_1)^j p_{t+1+j} \\
&\quad + \zeta\left(\frac{1-\beta\phi_1^2}{1-\beta\phi_1\rho_\xi}\frac{\hat{\kappa}_{l\xi}}{\hat{\kappa}_{ll}} - \frac{\beta\phi_1-\omega(1-\beta\phi_1)}{1-\beta\phi_1\rho_\xi}\frac{\psi}{\zeta}\right)\varepsilon_{\xi t+1}.
\end{aligned}$$

In terms of leverage and uncertainty we have

$$
\begin{aligned}
\frac{\omega\hat{\kappa}_{ll}}{\zeta}\delta'_{t+1} &= -\lambda\vartheta_l\Delta^e\mathbb{E}_{t+1}[\mu_{t+1}] + (\zeta-\psi)\,\Delta^e\mathbb{E}_{t+1}[u_{t+1}] + \omega\left(\hat{\kappa}_{l\xi} + \frac{\omega\psi}{\zeta}\hat{\kappa}_{ll}\right)\varepsilon_{\xi t+1} \\
&= -\lambda\vartheta_l\left(-\frac{(1+\omega)\varepsilon}{\beta\varphi_2\lambda}bM\Delta^e\mathbb{E}_{t+1}[w_{t+1}]\right) \\
&\quad + (\zeta-\psi)\left(-\frac{\beta\lambda\vartheta_l}{\zeta}\frac{(1+\omega)\varepsilon\phi_2}{\beta\varphi_2\lambda}bM\left(I - \left(\frac{\phi_2-\phi_1}{\phi_2-\varphi_1}\right)\left(I-\frac{1}{\phi_2}A\right)^{-1}\right)\Delta^e\mathbb{E}_{t+1}[w_{t+1}]\right) \\
&\quad + (\zeta-\psi)\left(-\frac{\phi_2 k}{\phi_2-\rho_\xi}\varepsilon_{\xi t+1}\right) + \omega\left(\hat{\kappa}_{l\xi} + \frac{\omega\psi}{\zeta}\hat{\kappa}_{ll}\right)\varepsilon_{\xi t+1} \\
&= \lambda\vartheta_l\frac{(1+\omega)\varepsilon}{\beta\varphi_2\lambda}\left(\frac{\phi_2-\phi_1}{\phi_2-\varphi_1}\right)bM\left(I-\frac{1}{\phi_2}A\right)^{-1}\Delta^e\mathbb{E}_{t+1}[w_{t+1}] \\
&\quad + \frac{\phi_2-\phi_1}{\phi_2-\rho_\xi}\omega\hat{\kappa}_{l\xi}\varepsilon_{\xi t+1} - \frac{1-\omega(\phi_2-1)}{\phi_2-\rho_\xi}\frac{\omega\psi}{\zeta}\hat{\kappa}_{ll}\varepsilon_{\xi t+1}
\end{aligned}
$$

Solving for $M\left(I-\frac{1}{\phi_2}A\right)^{-1}$ yields

$$
\begin{aligned}
M\left(I-\frac{1}{\phi_2}A\right)^{-1} &= \left(I-\frac{1}{\varphi_2}A\right)^{-1}\left(I-\frac{1}{\phi_2}A\right)^{-1} \\
&= \left(I-\frac{1}{\varphi_2}\begin{bmatrix}\phi_1 & -(1+\omega(1-\rho_\xi))\frac{\psi}{\zeta} \\ 0 & \rho_\xi\end{bmatrix}\right)^{-1}\left(I-\frac{1}{\phi_2}\begin{bmatrix}\phi_1 & -(1+\omega(1-\rho_\xi))\frac{\psi}{\zeta} \\ 0 & \rho_\xi\end{bmatrix}\right)^{-1} \\
&= \varphi_2\phi_2\begin{bmatrix}\frac{1}{(\varphi_2-\phi_1)(\phi_2-\phi_1)} & -\left(\frac{1}{\phi_2-\phi_1}+\frac{1}{\varphi_2-\rho_\xi}\right)\frac{(1+\omega(1-\rho_\xi))}{(\varphi_2-\phi_1)(\phi_2-\rho_\xi)}\frac{\psi}{\zeta} \\ 0 & \frac{1}{(\varphi_2-\rho_\xi)(\phi_2-\rho_\xi)}\end{bmatrix}.
\end{aligned}
$$

Using this, we can further simplify the expression for δ',

$$
\begin{aligned}
\frac{\omega\hat{\kappa}_{ll}}{\zeta}{\delta'}_{t+1} &= \lambda\vartheta_l \frac{(1+\omega)\varepsilon}{\beta\varphi_2\lambda}\left(\frac{\phi_2-\phi_1}{\phi_2-\varphi_1}\right) b\varphi_2\phi_2 \begin{bmatrix} \frac{1}{(\varphi_2-\phi_1)(\phi_2-\phi_1)} & -\frac{(\varphi_2-\rho_\xi+\phi_2-\phi_1)(1+\omega(1-\rho_\xi))}{(\phi_2-\phi_1)(\varphi_2-\rho_\xi)(\varphi_2-\phi_1)(\phi_2-\rho_\xi)}\frac{\psi}{\zeta} \\ 0 & \frac{1}{(\varphi_2-\rho_\xi)(\phi_2-\rho_\xi)} \end{bmatrix} \Delta^e \mathbb{E}_{t+1}[w_{t+1}] \\
&\quad + \frac{\phi_2-\phi_1}{\phi_2-\rho_\xi}\omega\hat{\kappa}_{l\xi}\varepsilon_{\xi t+1} - \frac{1-\omega(\phi_2-1)}{\phi_2-\rho_\xi}\frac{\omega\psi}{\zeta}\hat{\kappa}_{ll}\varepsilon_{\xi t+1} \\
&= \lambda\vartheta_l^2 \frac{(1+\omega)\varepsilon}{\beta}\left(\frac{\phi_2-\phi_1}{\phi_2-\varphi_1}\right)\phi_2 \frac{1}{(\varphi_2-\phi_1)(\phi_2-\phi_1)}\Delta^e\mathbb{E}_{t+1}[l_{t+1}] \\
&\quad + \lambda\vartheta_l^2 \frac{(1+\omega)\varepsilon}{\beta}\left(\frac{\phi_2-\phi_1}{\phi_2-\varphi_1}\right)\phi_2\left(-\frac{(\varphi_2-\rho_\xi+\phi_2-\phi_1)(1+\omega(1-\rho_\xi))}{(\phi_2-\phi_1)(\varphi_2-\rho_\xi)(\varphi_2-\phi_1)(\phi_2-\rho_\xi)}\frac{\psi}{\zeta}\right)\varepsilon_{\xi t+1} \\
&\quad + \lambda\vartheta_l\vartheta_\xi \frac{(1+\omega)\varepsilon}{\beta}\left(\frac{\phi_2-\phi_1}{\phi_2-\varphi_1}\right)\phi_2\frac{1}{(\varphi_2-\rho_\xi)(\phi_2-\rho_\xi)}\varepsilon_{\xi t+1} \\
&\quad + \frac{\phi_2-\phi_1}{\phi_2-\rho_\xi}\omega\hat{\kappa}_{l\xi}\varepsilon_{\xi t+1} - \frac{1-\omega(\phi_2-1)}{\phi_2-\rho_\xi}\frac{\omega\psi}{\zeta}\hat{\kappa}_{ll}\varepsilon_{\xi t+1} \\
&= \lambda\vartheta_l^2\frac{(1+\omega)\varepsilon}{\beta}\frac{\phi_2}{(\varphi_2-\phi_1)(\phi_2-\varphi_1)}\left(\frac{\omega\psi}{\zeta}\varepsilon_{\xi t+1} - \frac{1}{\zeta}{\delta'}_{t+1}\right) \\
&\quad - \lambda\vartheta_l^2\frac{(1+\omega)\varepsilon}{\beta}\frac{\phi_2(\varphi_2-\rho_\xi+\phi_2-\phi_1)(1+\omega(1-\rho_\xi))}{(\phi_2-\varphi_1)(\varphi_2-\rho_\xi)(\varphi_2-\phi_1)(\phi_2-\rho_\xi)}\frac{\psi}{\zeta}\varepsilon_{\xi t+1} \\
&\quad + \lambda\vartheta_l\vartheta_\xi\frac{(1+\omega)\varepsilon}{\beta}\frac{\phi_2(\phi_2-\phi_1)}{(\phi_2-\varphi_1)(\varphi_2-\rho_\xi)(\phi_2-\rho_\xi)}\varepsilon_{\xi t+1} \\
&\quad + \frac{\phi_2-\phi_1}{\phi_2-\rho_\xi}\omega\hat{\kappa}_{l\xi}\varepsilon_{\xi t+1} - \frac{1-\omega(\phi_2-1)}{\phi_2-\rho_\xi}\frac{\omega\psi}{\zeta}\hat{\kappa}_{ll}\varepsilon_{\xi t+1}
\end{aligned}
$$

$$
{\delta'}_{t+1} = \zeta\left(\frac{\chi\omega\hat{\kappa}_{l\xi}+(1+\omega)\vartheta_l\vartheta_\xi\varsigma(1-\gamma)}{\chi\omega\hat{\kappa}_{ll}+(1+\omega)\vartheta_l^2\varsigma}\left(\frac{\phi_2-\phi_1}{\phi_2-\rho_\xi}\right) - \frac{1-\omega(\phi_2-1)}{\phi_2-\rho_\xi}\frac{\psi}{\zeta}\right)\varepsilon_{\xi t+1} \tag{3.5}
$$

where

$$
\gamma = \frac{\phi_1-\rho_\xi+\frac{\vartheta_l}{\vartheta_\xi}(1+\omega(1-\rho_\xi))\frac{\psi}{\zeta}}{\varphi_2-\rho_\xi}
$$

$$
\varsigma = \frac{\lambda\chi\varepsilon}{\beta}\frac{\phi_2}{(\varphi_2-\phi_1)(\phi_2-\varphi_1)}
$$

F.3 Optimal macroprudential policy under sticky prices with an interest rate rule

In this section we'll derive optimal macroprudential policy under an interest rate rule regime. We'll focus on technology shocks only, which best illustrate the difference between this regime and the flexible price and optimal monetary policy regimes. Importantly, under an interest rate rule (or other, non-optimal monetary policy regimes) there is a role for macroprudential policy in reducing the welfare costs of fluctuations in marginal costs, even if those costs emerge from technology shocks.

We first seek a general solution to the following IS-PC system, derived from (1.1) and (1.2) with the interest rate rule $i_t = \phi_\pi \pi_t$:

$$
\begin{aligned}
x_t &= \mathbb{E}[x_{t+1}] - (\phi_\pi \pi_t - \mathbb{E}_t[\pi_{t+1}]) - \vartheta_l l_t \\
\pi_t &= \beta \mathbb{E}_t[\pi_{t+1}] + \lambda\chi(x_t - a_t) + \lambda\vartheta_l l_t
\end{aligned}
$$

Where $\vartheta := \frac{(\zeta-1)\psi}{\zeta}$. Guess and verify the following general solution:

$$x_t = \eta_{xa} a_t + \eta_{xl} l_t$$

$$\pi_t = \eta_{\pi a} a_t + \eta_{\pi l} l_t$$

The solution can be summarised as follows:

$$\eta_{xa} = -\left(\frac{\phi_\pi - \rho_a}{1-\rho_a}\right)\eta_{\pi a}, \qquad \eta_{xl} = -\left(\frac{\phi_\pi - \phi_1}{1-\phi_1}\right)\eta_{\pi l} - \frac{\vartheta_l}{1-\phi_1},$$

$$
\begin{aligned}
\eta_{\pi a} &= -\frac{(1-\rho_a)\lambda\chi}{(1-\beta\rho_a)(1-\rho_a) + (\phi_\pi - \rho_a)\lambda\chi}, \\
\eta_{\pi l} &= \frac{(1-\phi_1)\lambda\vartheta_l - \vartheta_l\lambda\chi}{(1-\beta\phi_1)(1-\phi_1) + (\phi_\pi - \phi_1)\lambda\chi}.
\end{aligned}
$$

We can treat this solution as a constraint on the macroprudential authority, summarising the IS-PC block of the model. The prudential policymaker's Lagrangian is

$$\mathcal{L} = \mathbb{E}\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{2}\left[(1+\omega)\left(\frac{\varepsilon}{\lambda}\pi_t^2+\chi\left(x_t^2-2x_ta_t\right)\right)+\omega\hat{\kappa}_{ll}l_t^2\right]\right.$$
$$-\mu_t\left[\pi_t-\eta_{\pi a}a_t-\eta_{\pi l}l_t\right]-\hat{\delta}'_t\left[x_t-\eta_{xa}a_t-\eta_{xl}l_t\right]$$
$$\left.-\nu_t\left[\zeta l_{t+1}-(\zeta-\psi)\,l_t\right]\right\},$$

The first order conditions are

$$\begin{aligned}
\pi: &\qquad 0=(1+\omega)\frac{\varepsilon}{\lambda}\pi_t-\mu_t\\
x: &\qquad 0=(1+\omega)\chi\left(x_t-a_t\right)-\hat{\delta}'_t\\
l: &\qquad 0=\omega\hat{\kappa}_{ll}l_t+\eta_{\pi l}\mu_t+\eta_{xl}\hat{\delta}'_t+(\zeta-\psi)\,\nu_t-\frac{\zeta}{\beta}\nu_{t-1}.
\end{aligned}$$

Use the optimality conditions to eliminate π, x from the IS-PC block:

$$\mu_t=(1+\omega)\frac{\varepsilon}{\lambda}(\eta_{\pi a}a_t+\eta_{\pi l}l_t)$$

$$\hat{\delta}'_t=(1+\omega)\chi\left((\eta_{xa}a_t+\eta_{xl}l_t)-a_t\right)$$

We can then use these expressions to eliminate $\mu, \hat{\omega}$ from the first order condition for leverage,

$$0=\omega\hat{\kappa}_{ll}l_t+\eta_{\pi l}(1+\omega)\frac{\varepsilon}{\lambda}(\eta_{\pi a}a_t+\eta_{\pi l}l_t)+\eta_{xl}(1+\omega)\chi\left((\eta_{xa}a_t+\eta_{xl}l_t)-a_t\right)+(\zeta-\psi)\,\nu_t-\frac{\zeta}{\beta}\nu_{t-1}.$$

$$0=\left(\omega\hat{\kappa}_{ll}+(1+\omega)\upsilon_l\right)l_t+(1+\omega)\upsilon_a a_t+(\zeta-\psi)\,\nu_t-\frac{\zeta}{\beta}\nu_{t-1}.$$

where

$$\upsilon_l:=\frac{\varepsilon}{\lambda}\eta_{\pi l}^2+\chi\eta_{xl}^2,\qquad \upsilon_a:=\frac{\varepsilon}{\lambda}\eta_{\pi l}\eta_{\pi a}+\chi\eta_{xl}(\eta_{xa}-1).$$

As in the flexible price or joint optimal policy regimes, after substituting this expression into the Leverage curve to eliminate leverage, we can solve for the following general solution for ν,

$$\nu_t=\phi_1\nu_{t-1}+\eta_a a_t$$

where $\phi_1 = \frac{\zeta-\psi}{\zeta}$. Substituting this general solution into the Leverage curve allows us to solve for η:

$$
\begin{aligned}
0 &= \zeta \mathbb{E}_t[l_{t+1}] - (\zeta - \psi)\, l_t \\
&= \zeta \mathbb{E}_t \left[-(1+\omega)\upsilon_a a_{t+1} - (\zeta-\psi)\,\nu_{t+1} + \frac{\zeta}{\beta}\nu_t \right] \\
&\qquad - (\zeta-\psi) \left[-(1+\omega)\upsilon_a a_t - (\zeta-\psi)\,\nu_t + \frac{\zeta}{\beta}\nu_{t-1} \right] \\
&= (\zeta-\psi)\,(\phi_2-\rho_a)\eta_a a_t + (\phi_1-\rho_a)(1+\omega)\upsilon_a a_t
\end{aligned}
$$

$$
\eta_a = -\left(\frac{\phi_1-\rho_a}{\phi_2-\rho_a}\right)\frac{(1+\omega)\upsilon_a}{\zeta-\psi}.
$$

Now, write the Leverage curve as

$$
0 = \zeta l_{t+1} - (\zeta-\psi)\, l_t + {\delta'}_{t+1},
$$

where ${\delta'}_{t+1}$ reflects macroprudential policy, and $\mathbb{E}_t[{\delta'}_{t+1}] = 0$.

$$
\begin{aligned}
0 &= \zeta l_{t+1} - (\zeta-\psi)\, l_t + {\delta'}_{t+1} \\
&= \zeta \left[-(1+\omega)\upsilon_a a_{t+1} - (\zeta-\psi)\,\nu_{t+1} + \frac{\zeta}{\beta}\nu_t \right] \\
&\qquad - (\zeta-\psi) \left[-(1+\omega)\upsilon_a a_t - (\zeta-\psi)\,\nu_t + \frac{\zeta}{\beta}\nu_{t-1} \right] + (\omega\hat{\kappa}_{ll} + (1+\omega)\upsilon_l)\,{\delta'}_{t+1}.
\end{aligned}
$$

Retain only the terms that are measurable in time $t+1$,

$$
\begin{aligned}
(\omega\hat{\kappa}_{ll} + (1+\omega)\upsilon_l)\,{\delta'}_{t+1} &= \zeta\left[(1+\omega)\upsilon_a + (\zeta-\psi)\,\eta_a\right]\varepsilon_{at+1} \\
&= \zeta\left(\frac{\phi_2-\phi_1}{\phi_2-\rho_a}\right)(1+\omega)\upsilon_a\varepsilon_{at+1}.
\end{aligned}
$$

Ultimately, we have

$$
{\delta'}_{t+1} = \zeta\left(\frac{\phi_2-\phi_1}{\phi_2-\rho_a}\right)\frac{(1+\omega)\left(\frac{\varepsilon}{\lambda}\eta_{\pi l}\eta_{\pi a} + \chi\eta_{xl}(\eta_{xa}-1)\right)}{\omega\hat{\kappa}_{ll} + (1+\omega)\left(\frac{\varepsilon}{\lambda}\eta_{\pi l}^2 + \chi\eta_{xl}^2\right)}\varepsilon_{at+1}.
$$

G Derivations for Section 4

The model

$$\vartheta_x x_t = -\vartheta_l l_t + \chi a_t + \gamma \varepsilon_{at}$$

$$l_t = \phi_l l_{t-1} - \phi_x (x_t - x_{t-1}) + \delta \varepsilon_{at}$$

Solving for leverage

$$\begin{aligned}
l_t &= \phi_l l_{t-1} - \phi_x (x_t - x_{t-1}) + \delta \varepsilon_{at} \\
\vartheta_x l_t &= \phi_l \vartheta_x l_{t-1} - \phi_x (-\vartheta_l l_t + \chi a_t + \gamma \varepsilon_{at}) + \phi_x (-\vartheta_l l_{t-1} + \chi a_{t-1} + \gamma \varepsilon_{at-1}) + \vartheta_x \delta \varepsilon_{at} \\
l_t &= \phi l_{t-1} - \frac{\phi_x}{\vartheta_x - \phi_x \vartheta_l} (\chi a_t + \gamma \varepsilon_{at}) + \frac{\phi_x}{\vartheta_x - \phi_x \vartheta_l} (\chi a_{t-1} + \gamma \varepsilon_{at-1}) + \frac{\vartheta_x}{\vartheta_x - \phi_x \vartheta_l} \delta \varepsilon_{at}
\end{aligned}$$

Iterating backward

$$\begin{aligned}
l_t = &- \frac{\phi_x}{\vartheta_x - \phi_x \vartheta_l} (\chi \delta a_t + \gamma \delta \varepsilon_{at}) + \frac{\vartheta_x}{\vartheta_x - \phi_x \vartheta_l} \delta \varepsilon_{at} + \phi \left[- \frac{\phi_x}{\vartheta_x - \phi_x \vartheta_l} (\chi \delta a_{t-1} + \gamma \delta \varepsilon_{at-1}) + \frac{\vartheta_x}{\vartheta_x - \phi_x \vartheta_l} \delta \varepsilon_{at-1} \right] \\
&+ \phi^2 \left[- \frac{\phi_x}{\vartheta_x - \phi_x \vartheta_l} (\chi \delta a_{t-2} + \gamma \delta \varepsilon_{at-2}) + \frac{\vartheta_x}{\vartheta_x - \phi_x \vartheta_l} \delta \varepsilon_{at-2} \right] \\
&+ \phi^3 \left[- \frac{\phi_x}{\vartheta_x - \phi_x \vartheta_l} (\chi \delta a_{t-3} + \gamma \delta \varepsilon_{at-3}) + \frac{\vartheta_x}{\vartheta_x - \phi_x \vartheta_l} \delta \varepsilon_{at-3} \right] + ... \\
= &- \frac{\phi_x}{\vartheta_x - \phi_x \vartheta_l} (\chi \varepsilon_{at} + \gamma \varepsilon_{at}) + \frac{\vartheta_x}{\vartheta_x - \phi_x \vartheta_l} \delta \varepsilon_{at}
\end{aligned}$$

$$\begin{aligned}
l_t = &- \frac{\phi_x \chi}{\vartheta_x - \phi_x \vartheta_l} \varepsilon_{at} - \frac{\phi_x \chi}{\vartheta_x - \phi_x \vartheta_l} \left[\rho - 1 + \phi \right] \varepsilon_{at-1} - \frac{\phi_x \chi}{\vartheta_x - \phi_x \vartheta_l} \left[\rho^2 - \rho + \phi \rho - \phi + \phi^2 \right] \varepsilon_{at-2} \\
&- \frac{\phi_x \chi}{\vartheta_x - \phi_x \vartheta_l} \left[\rho^3 - \rho^2 + \phi \rho^2 - \phi \rho + \phi^2 \rho - \phi^2 + \phi^3 \right] \varepsilon_{at-3} - ... \\
&- \frac{\phi_x \gamma}{\vartheta_x - \phi_x \vartheta_l} \varepsilon_{at} + \frac{(1 - \phi) \phi_x \gamma}{\vartheta_x - \phi_x \vartheta_l} \sum_{\tau=1}^{\infty} \phi^{\tau - 1} \varepsilon_{at-\tau} + \frac{\vartheta_x \delta}{\vartheta_x - \phi_x \vartheta_l} \sum_{\tau=0}^{\infty} \phi^{\tau} \varepsilon_{at-\tau} + ...
\end{aligned}$$

$$l_t = -\frac{\phi_x \chi}{\vartheta_x - \phi_x \vartheta_l} \sum_{\tau=0}^{\infty} \phi^{\tau} \varepsilon_{at-\tau} + (1-\rho) \frac{\phi_x \chi}{\vartheta_x - \phi_x \vartheta_l} \varepsilon_{at-1} + (1-\rho) \frac{\phi_x \chi}{\vartheta_x - \phi_x \vartheta_l} [\rho + \phi] \varepsilon_{at-2}$$
$$+ (1-\rho) \frac{\phi_x \chi}{\vartheta_x - \phi_x \vartheta_l} \left[\rho^2 + \phi\rho + \phi^2\right] \varepsilon_{at-3} + ...$$
$$- \frac{\phi_x \gamma}{\vartheta_x - \phi_x \vartheta_l} \varepsilon_{at} + \frac{(1-\phi)\phi_x \gamma}{\vartheta_x - \phi_x \vartheta_l} \sum_{\tau=1}^{\infty} \phi^{\tau-1} \varepsilon_{at-\tau}$$
$$+ \frac{\vartheta_x \delta}{\vartheta_x - \phi_x \vartheta_l} \sum_{\tau=0}^{\infty} \phi^{\tau} \varepsilon_{at-\tau}$$

$$l_t = -\frac{\phi_x \chi}{\vartheta_x - \phi_x \vartheta_l} \sum_{\tau=0}^{\infty} \phi^{\tau} \varepsilon_{at-\tau} + \frac{\phi_x \chi}{\vartheta_x - \phi_x \vartheta_l} \sum_{\tau=1}^{\infty} \left((1-\rho) \sum_{j=1}^{\tau} \phi^{j-1} \rho^{\tau-j} \right) \varepsilon_{at-\tau}$$
$$- \frac{\phi_x \gamma}{\vartheta_x - \phi_x \vartheta_l} \sum_{\tau=0}^{\infty} \phi^{\tau} \varepsilon_{at-\tau} + \frac{\phi_x \gamma}{\vartheta_x - \phi_x \vartheta_l} \sum_{\tau=1}^{\infty} \phi^{\tau-1} \varepsilon_{at-\tau}$$
$$+ \frac{\vartheta_x \delta}{\vartheta_x - \phi_x \vartheta_l} \sum_{\tau=0}^{\infty} \phi^{\tau} \varepsilon_{at-\tau}$$

Solving for output

We start by re-writing the Phillips curve as follows:

$$x_t = -\frac{\vartheta_l}{\vartheta_x} l_t + \frac{1}{\vartheta_x} \chi a_t + \frac{1}{\vartheta_x} \gamma \varepsilon_{at}$$

Substituting the solution for leverage yields:

$$x_t = \frac{\chi}{\vartheta_x} \frac{\vartheta_l \phi_x}{\vartheta_x - \phi_x \vartheta_l} \left[\sum_{\tau=0}^{\infty} \phi^{\tau} \varepsilon_{at-\tau} - (1-\rho) \sum_{\tau=1}^{\infty} \sum_{j=1}^{\tau} \phi^{j-1} \rho^{\tau-j} \varepsilon_{at-\tau} \right] + \frac{\chi}{\vartheta_x} \sum_{\tau=0}^{\infty} \rho^{\tau} \varepsilon_{at-\tau}$$
$$+ \frac{\gamma}{\vartheta_x} \frac{\vartheta_l \phi_x}{\vartheta_x - \phi_x \vartheta_l} \left[\sum_{\tau=0}^{\infty} \phi^{\tau} \varepsilon_{at-\tau} - \sum_{\tau=1}^{\infty} \phi^{\tau-1} \varepsilon_{at-\tau} \right] + \frac{\gamma}{\vartheta_x} \varepsilon_{at}$$
$$- \frac{\vartheta_l \delta}{\vartheta_x - \phi_x \vartheta_l} \sum_{\tau=0}^{\infty} \phi^{\tau} \varepsilon_{at-\tau}$$

H Derivations for Section 5

Uncertainty shocks

Start by iterating (5.4) backward,

$$
\begin{aligned}
x_t &= x_{t-1} + \left(\frac{\zeta + \sigma\omega\psi}{\sigma - 1}\right)(\xi_t - \xi_{t-1}) \\
&= \left(\frac{\zeta + \sigma\omega\psi}{\sigma - 1}\right)(\xi_t - \xi_{t-1} + (\xi_{t-1} - \xi_{t-2}) + (\xi_{t-2} - \xi_{t-3})...)
\end{aligned}
$$

$$
x_{t+\tau} = \left(\frac{\zeta + \sigma\omega\psi}{\sigma - 1}\right)\xi_{t+\tau} \quad \forall \tau \geq 0. \tag{5.9}
$$

Now, substitute this condition into the Phillips curve and iterate forward,

$$
\begin{aligned}
\pi_t &= \beta \mathbb{E}_t[\pi_{t+1}] + \lambda(\sigma + \chi - 1)x_t + \lambda(\theta_\xi - \theta_l - \sigma\omega)\xi_t \\
&= \beta \mathbb{E}_t[\pi_{t+1}] + \lambda \underbrace{\left[(\sigma + \chi - 1)\left(\frac{\zeta + \sigma\omega\psi}{\sigma - 1}\right) + (\theta_\xi - \theta_l - \sigma\omega)\right]}_{:=\mu_\xi}\xi_t \\
&= \beta \mathbb{E}_t[\pi_{t+1}] + \lambda\mu_\xi\xi_t
\end{aligned}
$$

$$
\pi_{t+\tau} = \frac{\lambda\mu_\xi}{1 - \beta\rho_\xi}\xi_{t+\tau} \quad \forall \tau \geq 0.
$$

Solving for path of real interest rates using (5.5) yields

$$
\begin{aligned}
r_t &= \left(1 + \frac{\sigma - 1}{\zeta}\right)(\mathbb{E}[x_{t+1}] - x_t) + \frac{\sigma\omega\psi}{\zeta}(1 - \rho_\xi)\xi_t \\
&= -\left(1 + \frac{\sigma - 1}{\zeta}\right)\left(\frac{\zeta + \sigma\omega\psi}{\sigma - 1}\right)(1 - \rho_\xi)\xi_{t+\tau} + \frac{\sigma\omega\psi}{\zeta}(1 - \rho_\xi)\xi_t \\
&= -\left(\frac{\zeta + \sigma - 1 + \sigma\omega\psi}{\sigma - 1}\right)(1 - \rho_\xi)\xi_t
\end{aligned}
$$

$$r_{t+\tau} = -\left(\frac{\sigma(1+\omega)}{\sigma - 1}\right)(1-\rho_\xi)\xi_{t+\tau} \quad \forall \tau \geq 0.$$

References

Guillermo A. Calvo. Staggered prices in a utility-maximizing framework. *Journal of Monetary Economics*, 12(3):383–398, 1983. ISSN 0304-3932.

Alfred Duncan and Charles Nolan. Disputes, debt and equity. *Theoretical Economics*, 14(3), September 2019.

Jordi Galí. *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework*. Princeton University Press, June 2008.

Takashi Negishi. Welfare economics and existence of an equilibrium for a competitive economy. *Metroeconomica*, 12(23):92–97, 1960. doi: 10.1111/j.1467-999X.1960.tb00275.x.

M. Woodford. *Interest and prices: Foundations of a theory of monetary policy*. Princeton University Press, 2003.